In the last few years it has become usual to invert the distribution function transform to obtain the probability elements involved in option prices. Heston, Bakshi-Cao-Chen, Bates and Carr-Geman-Madan-Yor among others use this procedure. The method has an inconvenient: the integrand involved has a singularity at the origin, so the Fast Fourier Transform can not be applied.

Besides, a deviation from existent literature (based on the arbitrage approach) has arisen (Geman-Madan-Yor). Under this new perspective, prices are typically expressed as Lévy processes and the continuity and normality can be recovered by means of a time change. The time thus considered is random, representing a measure of the economic activity.

This paper is concerned about both discoveries. For models where the time is random, we develop a method to obtain expressions computable by means of the Fast Fourier Transform. The method, based on complex variable theory, exploits the numerical restrictions imposed on the model parameters by the “convexity correction”.

Firstly, we analyse the Variance Gamma model. An alternative method to derive the density function is proposed, as well as an expression for the European call computable by FFT. This expression also permits pricing options accurately by interpolation of elementary functions.

The method can be generalised to a wide class of processes useful in finance. The processes are those in the generalised gamma convolution, obtaining expressions computable by FFT. Some important examples like the CGMY and the CMY models are provided.
Although the Black-Scholes model remains as the most extended option pricing model, it has well known limitations broadly investigated\(^1\). These limitations are in connection to the central role played by the Brownian motion in the model, because the *gaussian* assumption contradicts the markets reality in several ways.

The biases may be explained in terms of the superior moments of the underlying price distributions, clearly apart from the *normality* postulated by Black-Scholes’73 (BS further on). Such distributional features have been studied using time series of stock and option markets, characterising the last ones the so-called volatility *smile*.

Some important alternatives proposed in the literature are the jump-diffusion models, the stochastic volatility models and models based on Lévy processes (with finite or infinite arrival rate of jumps) like the Variance Gamma model.

The Merton model\(^2\) introduce stock price discontinuities, because continuity of pure diffusions is actually a drawback. The resultant model leads to non-exact expressions for pricing. Besides, the high number of parameters (five at the minimum) complicates its use\(^3\). Moreover, as pointed out in McCulloch’78, it does not provide the inherent cohesion of other jump models.

Jump-diffusion models capture relatively well the *smile* shown by the short dated options (Das-Sundaram’98 or Bates ’96)

In the stochastic volatility models, the volatility follows a diffusion (Hull-White’87 or Heston’93a among others) These models are able to capture the smile, but just qualitatively. They always fail in explaining its magnitude. Bates’96 shows that the mean-reverting volatility *submodel* cannot explain the smile evidence of implicit excess kurtosis except under implausible parameters.

Why are stochastic volatility models unable to explain the markets reality? Empirically it has been noted that as sampling period decreases, the kurtosis of the returns increases. That is just opposite to what happens in the models where the volatility is driven by a diffusion. Locally, diffusion models are *gaussian*, in sharp contrast to reality. So for short dated options, this kind of models is expected to work badly. That is the reason of introducing a jump component in modeling stock price dynamics, as it is noted in Bakshi-Cao-Chen’97.

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\(^1\) See Rubinstein (’85, ’94), for example.

\(^2\) Merton’76 is the first of this kind of models, contemporary to Cox-Ross’ pure-jump models.

\(^3\) Standard Maximum Likelihood procedures in estimating jump-diffusion models are no longer valid (Honoré ’98)
For a detailed analysis of the term structure of the volatility smile, see Das-Sundaram’98. Another drawback concerning to both kinds of models is pointed out in Bates’96. The stochastic volatility/jump-diffusion model analysed exhibits parameter instability. This is a consequence of its infinite variation. As Madan’99 points out, the lack of robustness of diffusions is an inherent property of infinite variation processes\(^4\), and he strongly advocates against the use of this kind of processes as models of price dynamics.

Finally we have the VG model and in general pure-jump models with finite or infinite arrival rates (the VG belongs to the first class). The corresponding logprice processes are infinitely divisible with independent increments, so they may be described by their Lévy-Khintchine densities. The hyperbolic model (Eberlein-Keller-Prause’98) is another example, but it has several disadvantages if compared to the VG model.

The Variance Gamma model is a three-parameter one that eliminates the volatility smile in the strike direction. Its pricing accuracy lies generally between 1 and 3 per cent, considering strikes up to 20 to 30% out-the-money and maturities as short as two days sometimes (see Madan’99).

The density function and the call price may be expressed as closed forms in terms of special functions, and the parameter estimation may be performed by Maximum Likelihood.

The formulas derived for the VG model have a disadvantage: the functions involved, expressed as power series, are computationally expensive. To price options using the closed form may be slower than computing prices by numerical procedures employing Fourier-inversion\(^5\). Some drawbacks of these numerical methods are commented on below.

The objective of this paper is to overcome these limitations, providing an alternative method to obtain simple formulas.

By means of complex variable arguments, we will firstly provide an alternative derivation of the VG logprice density function. Then, we will obtain an alternative integral form for the European call. It will be an integral expression of elementary functions that may be directly performed by FFT, making unnecessary to introduce external factors like in Carr-Madan’98.

This new formulation has other advantages. It makes possible to value options by interpolation. This possibility for option pricing is accurate, being low its computational cost because elementary functions are involved.

Furthermore, an alternative closed form simpler than the one proposed in Madan-Carr-Chang’98 can be obtained.

The VG model represents a particular case of a wide class of processes. The method is generalised and some important examples are provided.

The rest of the paper is structured as follows. In the next section we introduce the VG model. In Section 2 the density function and the call price are obtained employing an alternative method based on complex variable theory. The VG prices are interpolated in Section 3. Section 4 generalises the method, providing expressions for some useful pure-jump models. Section 5 presents the main conclusions.

\(^4\) A process of finite variation requires that the integral \(\int_{0}^{\infty} x k(x) \, dx\) be finite. In such processes, the sum of their moves in absolute magnitude is finite.

\(^5\) See Carr-Madan’98.
1.- THE VG MODEL

The VG model generalises the Brownian Motion as logprice process and it is explained in its general form in Madan-Carr-Chang ’98. That paper generalises the two previous formulations: Madan-Seneta ’90 considers symmetric returns and Madan-Milne’91 is intended to option pricing, by skewing the former in a general equilibrium setting.

In the VG model, the statistical and the risk-neutral price dynamics are given in terms of the VG process. The price dynamics will be precisely described below, after defining and characterising the VG process.

The VG process is a pure jump model, and their three parameters $\sigma, \theta, \nu$ take into account the variance, skewness and kurtosis of the price process.

Specifically, it is obtained as a Brownian motion (BM) with drift evaluated at a random time $\gamma(t)$:

$$X_t = \theta \gamma(t) + \sigma W_{\gamma(t)}$$ [0]

being $W_t$ a standard BM and $\gamma(t)$ a gamma process evaluated at $t$. The BM requires no further explanation. The gamma process is a infinitely divisible one, obtained by adding independent increments which follow a gamma random variable. The density function of a random gamma variable of parameters $(a, p)$ is given by:

$$f_{\gamma(t)}(x) = \frac{a^p}{\Gamma(p)} x^{p-1} e^{-ax}$$ [1]

So $\gamma(t)$ is a non-decreasing process distributed as a gamma random variable of parameters $a = 1/\nu, p = 1/\nu$, and may be approximated as a compound Poisson process.

It may also be described by means of its characteristic function, univocally obtained by the inverse Fourier transform of the density function given above:

$$\phi_{\gamma(t)}(u) = \left( \frac{1}{1 - iuv / \mu} \right)^{i/\nu}$$ [2]

As infinitely divisible process, it may also be characterised by means of its Lévy measure:

$$k_{\gamma}(x)dx = \begin{cases} \mu^2 \frac{e^{-x}}{x} dx, x > 0 \\ 0, x \leq 0 \end{cases}$$

The integral of this function is infinite, so the gamma process has infinite activity. As $k(0)$ is also infinite, the measure is concentrated near the origin. The coefficient of the diffusion term $-u^2/2$ in the Lévy-Khintchine representation of its characteristic function is zero, and then the process is a pure-jump one.

By evaluating a BM with drift at a gamma random time, we obtain the VG process. Its density function is not as simple as in the gamma process, but its characteristic function and Lévy density are.

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6 This process is employed by Heston’93b for modeling price dynamics.
The characteristic function may be evaluated integrating the Brownian characteristic function with respect to the variance gamma measure \( f_{\gamma(t)}(x)dx, f_{\gamma(t)}(x) \) given above. It yields to the simple expression:

\[
\phi_{VG}(u) = \left(\frac{1}{1 - i\theta v u + (\sigma^2 v/2)u^2}\right)^{1/v} \tag{3}
\]

The notion of equilibrium is implicit in the model, as it is explained in Geman-Madan-Yor ’98 or Madan‘99 and it may be summarised as follows: the VG model is obtained by evaluating the BM at a random time, and in Monroe ’78 it is demonstrated that it implies to be a semimartingale\(^7\). And a price process is a semimartingale if and only if No-Free-Lunch-with-Vanishing-Risk is possible (Delbaen-Schachermayer’94). This condition is more restrictive than the No-Arbitrage condition. Moreover, as no particular form has been assigned to the risk premium (the price process can be derived from a Lucas-type economy) the equilibrium is general and not only partial.

This simple expression allows us to express the VG process in an alternative form. It consists of expressing it as the difference of two gamma processes.

Factoring the characteristic function, it may be expressed as:

\[
\phi_{VG}(u) = \left(\frac{1}{1 - \frac{i\nu}{\mu_p}u} \right)^{\nu_p} \left(\frac{1 + \frac{i\nu}{\mu_p}u}{1 + \frac{i\nu}{\mu_p}u}\right)^{\nu_p},
\]

The parameters \(\mu_p, \mu_n, \nu_p, \nu_n\) may be related to the original ones (see Madan-Carr-Chang ’98)

As the gamma process is non-decreasing, the price process can be viewed as the difference of two gamma processes. They take account of the up and down moves of logprices. The VG process is consequently of finite variation\(^8\), which may also be shown by integrating \(xk(x)\) in a neighbourhood of zero.

The VG Lévy-Khintchine density is simply given by:

\[
k_{VG}(x)dx = \begin{cases} \\
\frac{\mu_n^2 e^{\frac{\mu_n|x|}{\nu_n}}}{|x|} dx, x < 0 \\
\frac{\mu_p^2 e^{\frac{\mu_p|x|}{\nu_p}}}{|x|} dx, x > 0
\end{cases}
\]

The VG has been described and characterised. Now we introduce the statistical and the risk-neutral price dynamics.

The statistical price dynamics is given by

\(^7\) If the price process is a semimartingale, there exists a probability measure under which price processes are martingales.

\(^8\) As it has already been commented on, it leads to a better tolerance of parametric heterogeneity.
\[ S_t = S_0 \exp \left( mt + \omega t + X_t(\sigma, \theta, \nu) \right) \]  \[ \text{[4]} \]

where \( X_t \) is a VG process, \( m \) is the statistical mean return and \( \omega \) is the convexity correction calculated by evaluating the characteristic function at \(-i^9\), in this case resulting:

\[ \omega = \frac{1}{\nu} \log(1 - \theta \nu - \sigma^2 \nu / 2) \]  \[ \text{[5]} \]

It introduces a restriction on the feasible parameters of the model, because \( \omega \) must be real. This has not been sufficiently pointed out before and it will be used along the paper.

The density function can be obtained by integrating the \textit{gaussian} density with respect the random time density function. It leads to an expression containing the \textit{McDonald function} (or \textit{modified Bessel function of the second kind}):

\[
f_{X_t}(x) = \frac{2e^{\alpha x/\sigma^2}}{\nu^{1/\nu} \sqrt{2\pi \sigma^2 \Gamma(\nu/2)}} \left( \frac{x^2}{2\sigma^2 / \nu + \theta^2} \right)^{t/2} K_{t-1/2} \left( \frac{\sqrt{2\sigma^2 / \nu + \theta^2}}{\sigma^2} x \right) \]  \[ \text{[6]} \]

This returns density permits to test how the model fits stock market prices. The risk-neutral price dynamics (the relevant one in futures and options markets) is obtained in a completely similar manner. It just requires to substitute the statistical parameters by their risk-neutral counterparts. As it is well known, the risk-neutral counterpart to the statistical drift \( m \) is the interest rate \( r \), because discounted stock prices under risk-neutral valuation have to be martingales.

The European call price is then obtained by calculating the expectation \( e^{-rT} E^{RN}(S_T - K)_+ \). This expectation is taken under the risk-neutral probability, so employing risk-neutral parameters. This involves integrating the BS formula with respect the gamma density. This is the method employed in Madan-Carr-Chang’98. There are other possibilities. One of them is to compute the integral by using the returns density function obtained above. Another is the method proposed in this paper, using the VG characteristic function and complex variable theory. Other possibilities are commented below.

The European call price obtained in Madan-Carr-Chang’98 is expressed in closed-form\(^{10}\) in terms of special functions given by:

\[ C_T = S_0 \psi \left( d \sqrt{\frac{c_1}{c_2}}, (\alpha + s) \sqrt{\frac{v}{c_1}}, \gamma \right) - Ke^{-rT} \psi \left( d \sqrt{\frac{c_1}{c_2}}, \alpha \sqrt{\frac{v}{c_1}}, \gamma \right), \]  \[ \text{[7]} \]

where \( s, \alpha, \gamma, c_1, c_2, d \) are functions of \( \sigma, \theta, \nu \).

The function \( \psi \) may be expressed in terms of special functions and is given by:

\(^9\) \( E^{RN}(S_T) = e^{rT} S_0 \Rightarrow E^{RN}(e^{X_T}) = e^{-\nu T} \Rightarrow \phi(-i) = e^{-\nu T} \)

\(^{10}\) In the referred paper is explained what is understood as a “closed form”.

6
\[ \psi(a,b;\gamma) = \frac{c^{\gamma+\frac{1}{2}}e^{\text{sign}(a)\kappa} (1+u)^\gamma}{\sqrt{2\pi \Gamma(\gamma \gamma)}} K_{\gamma+\frac{1}{2}}(c)\Phi(\gamma,1-\gamma,1+\gamma;\frac{i\mu}{2},-\text{sign}(a)c(1+u)) - \\
- \text{sign}(a) \frac{c^{\gamma+\frac{1}{2}}e^{\text{sign}(a)\kappa} (1+u)^\gamma}{\sqrt{2\pi \Gamma(\gamma)(1+\gamma)}} K_{\gamma-\frac{1}{2}}(c)\Phi(1+\gamma,1-\gamma,2+\gamma;\frac{i\mu}{2},-\text{sign}(a)c(1+u)) + \\
+ \text{sign}(a) \frac{c^{\gamma+\frac{1}{2}}e^{\text{sign}(a)\kappa} (1+u)^\gamma}{\sqrt{2\pi \Gamma(\gamma \gamma)}} K_{\gamma-\frac{1}{2}}(c)\Phi(\gamma,1-\gamma,1+\gamma;\frac{i\mu}{2},-\text{sign}(a)c(1+u)) \]  

where: \( c = \sqrt{2+b^2} \), \( u = \frac{b}{\sqrt{2+b^2}} \) and \( K_a(x) \) and \( \Phi(\alpha,\beta;\gamma;x,y) \) represent the modified Bessel function of the second kind and the degenerate hypergeometric function of two variables respectively.

This closed form has a disadvantage. The functions involved, expressible as power series, are computationally expensive. Calculating the call price using [7] is slower\(^{12}\) than computing the price numerically. For it, Carr-Madan’98 transform that integral in such a way that option prices may be performed by FFT. It leads to a much faster solution than using the closed form [7]. The method has the disadvantage that it needs an external dampening factor to employ the FFT.

The present article fills this gap by providing an alternative method to obtain the formulas. The pricing expression is obtained as an integral of an elementary function that can be performed by FFT, making unnecessary introducing a dampening factor.

The VG processes and in general truncated Lévy flights slowly converge towards the Gaussian process (Koponen’95). This is in connection to that BS prices better long-dated options.

2.- THE ALTERNATIVE METHOD

Let us consider a VG gamma process given by \( X_t = \theta \gamma(t) + \sigma W_{\gamma(t)} \).

The characteristic function is given in [3]:

\[ \phi_{VG}(u) = \left( \frac{1}{1-i\theta u + (\sigma^2 \gamma^2 / 2)u^2} \right)^{i/2} \]

It is well known that the VG density can be expressed as the inverse Fourier transform of \( \phi_{VG}(u) \):

\[ f_{X_t}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \left( \frac{1}{1-i\theta v + (\sigma^2 \gamma^2 / 2)v^2} \right)^{i/2} dv \]

\(^{11}\) Also represented as \( F_1(\alpha,\beta;\gamma;x,y) \equiv M(\alpha,\beta;\gamma;x,y) \), one of the confluent hypergeometric functions.

\(^{12}\) See Carr-Madan’98.
an alternative to integrating the *gaussian* characteristic function with respect the gamma-distributed variance [1].

This integral may be easily expressed as a real form using complex variable theory, obtaining the VG-density function [6].

We first make the change of variable $z = u - i\theta / \sigma^2$, resulting:

$$f_{X_t}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\theta x} \left( \frac{1}{1 - i\theta \nu u + (\sigma^2 \nu / 2) u^2} \right)^{\gamma/\nu} du =$$

$$\frac{1}{2\pi} \left( \frac{1}{\sigma^2 \nu / 2} \right)^{\gamma/\nu} \int_{-\infty}^{\infty} e^{-\theta x / \sigma^2} \left( \frac{1}{u^2 - (2i\theta / \sigma^2) u + 2/\sigma^2} \right)^{\gamma/\nu} du =$$

$$\frac{1}{2\pi} \left( \frac{1}{\sigma^2 \nu / 2} \right)^{\gamma/\nu} \theta e^{\theta x / \sigma^2} \int_{-\infty}^{\infty} e^{-\theta x / \sigma^2} \left( \frac{1}{z^2 + \theta^2 / \sigma^4 + 2/\sigma^2} \right)^{\gamma/\nu} \, dz =$$

$$\frac{1}{2\pi} \left( \frac{1}{\sigma^2 \nu / 2} \right)^{\gamma/\nu} \theta e^{\theta x / \sigma^2} \int_{-\infty}^{\infty} e^{-\theta x / \sigma^2} \left( \frac{1}{z^2 + \sigma^2/\nu} \right)^{\gamma/\nu} \, dz, \text{ making } \frac{\sigma^2}{\nu} + \frac{\sigma^2}{\nu} = \beta^2$$

We now consider an infinite rectangle in $C$, consisting of the real axis, a parallel axis given by $R - i\theta / \sigma^2$ and two symmetrical vertical segments closing the path at the infinite. The integrand is a complex function, and we now enquire where is it analytical. This may be easily checked it out by applying the Cauchy-Riemann conditions, resulting that:

- If $t/\nu$ is an integer, the integrand is analytical if $z \neq +i\beta, -i\beta$. This case will have interesting consequences commented below. But generally, $t$ must not be considered as an integer, because it represents the time parameter.
- If $t$ is real but not an integer, the function under consideration will be analytical if and only if $z^2 + \beta^2$ is positive.

So the integrand is analytical in $C$ excepting in the set \{ $i|x|, x \in R, |x| \geq \beta$ \}

As $\frac{\sigma^2}{\nu} < \beta^2$, the rectangle in consideration lies in the region where the integrand is analytical. By application of the Cauchy’s theorem13:

$$\int_{-\infty}^{\infty} \frac{e^{-ix/\sigma^2}}{\sigma^2} \left( \frac{1}{z^2 + \beta^2} \right)^{\gamma/\nu} \, dz = \int_{-\infty}^{\infty} \frac{e^{-ix/\sigma^2}}{\sigma^2} \left( \frac{1}{z^2 + \sigma^2/\nu} \right)^{\gamma/\nu} \, dz,$$

taking into account that the other two contributions are zero.

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13 If $f : G \rightarrow C$ is an analytical function and $\gamma$ is a closed rectifiable curve in $G$ such $\gamma$ is homotopic to zero, then

$$\int_{\gamma} f = 0 \ (\text{Conway’78})$$
This resulting integral expression can be written in terms of the modified Bessel function of the second kind as (see Gradshteyn and Ryzhik’65, pg.959):

\[
\int_{-\infty}^{+\infty} e^{-izx} \left[ \frac{1}{z^2 + \beta^2} \right]^{1/2} \, dz = \frac{2\sqrt{\pi}}{2^{1/4} \Gamma(t/v)} \left( \frac{x}{\beta} \right)^{1/2} K_{1/2}(\beta x), \text{ which leads to } [5]
\]

Now we are interested in obtaining an alternative expression for the European call price. We will perform the calculation following the same line of reasoning used above, obtaining a fairly simple integral expression.

For the sake of simplicity and for reasons of economic relevance, we will restrict ourselves to the case \( \theta < 0 \). It does not imply a lost of generality, because a parallel derivation can be performed for \( \theta > 0 \). But \( \theta < 0 \) corresponds to the case of positive risk aversion in options markets, the relevant case. It causes the smile asymmetry and is in agreement with empirical studies. All the cases studied in Carr-Geman-Madan-Yor’00 relative to option markets are negatively skewed with negative \( \theta \).

We have then to calculate \( C_T = e^{-rT}E^{RN} (S_T - K) \), where \( RN \) denotes that the expectation is taken under the risk-neutral probability:

\[ C_T = e^{-rT}E(S_T - K)_+ = E(S_0e^{\theta TX_t} - e^{-rT}K)_+ \text{ being } X_t \text{ the risk-neutral VG process and } e^{-\theta T} = \phi(-i) \text{ the convexity correction.} \]

\[ C_T = S_0e^{\theta T} \int_B e^x f_{X_t}(x) \, dx - Ke^{-rT} \int_B f_{X_T}(x) \, dx \text{ and } B = \log(K/S_0) - rT - \omega T. \]

Both integrals belong to the same parametric class. Let us see it.

Let us denote \( \int_B f_{X_T}(x) \, dx \) by \( I(1/v, \theta, T; B) \).

Using [1], \( \int_B e^x f_{X_T}(x) \, dx \) may be expressed as

\[
\int_B dx e^{x} \int_0^\infty \frac{1}{\sigma \sqrt{2\pi \tau}} e^{\frac{(x-\theta \tau)^2}{2\sigma^2 \tau}} \frac{\tau^{T/v-1} e^{-\tau/v}}{v^{T/v} \Gamma(T/v)} d\tau =
\]

\[
\int_B dx \int_0^\infty \frac{1}{\sigma \sqrt{2\pi \tau}} e^{\frac{(x-\theta + \sigma \tau)^2}{2\sigma^2 \tau}} \frac{\tau^{T/v-1} e^{-\tau/v}}{v^{T/v} \Gamma(T/v)} d\tau =
\]

\[
\int_B dx \int_0^\infty \frac{1}{\sigma \sqrt{2\pi \tau}} e^{\frac{(x-\theta - \sigma \tau)^2}{2\sigma^2 \tau}} \frac{\tau^{T/v-1} e^{-\tau/v}}{v^{T/v} \Gamma(T/v)} d\tau
\]
which expressed in terms of \( I(\cdot) \) results\(^{14}\):

\[
\int_{B}^{\infty} e^{z} f_{X_{2}}(x) dx = \left( \frac{1}{\sqrt{2\pi}} \right)^{T} I\left( \frac{1}{V} - \frac{\alpha^{2}}{2}, \theta + \sigma^{2}, T; B \right) = e^{-\alpha T} I\left( \frac{1}{V} - \frac{\alpha^{2}}{2}, \theta + \sigma^{2}, T; B \right)
\]

and then the call price will be, with the restrictions commented above:

\[
C_{v} = S_{0} I\left( \frac{1}{V} - \frac{\alpha^{2}}{2}, \theta + \sigma^{2}, T; B \right) - K e^{-rT} I\left( \frac{1}{V}, \theta, T; B \right)
\] \hspace{1cm} [9a]

Now, we will obtain a simple integral expression for \( I(1/V, \theta, T; B) \). It will be based on the VG logprice density, before expressing it in closed form.

\[
I(1/V, \theta, T; B) = \int_{B}^{\infty} f_{X_{2}}(x) dx = \frac{1}{2\pi} \left( \frac{1}{\sigma^{2} / 2} \right)^{T/V} \int_{B}^{\infty} dx \sigma^{-x} \int_{-\infty}^{\infty} \left( \frac{1}{z^{2} + \beta^{2}} \right)^{T/V} dz =
\]

\[
= \frac{1}{2\pi} \left( \frac{1}{\sigma^{2} / 2} \right)^{T/V} \int_{-\infty}^{\infty} \left( \frac{1}{z^{2} + \beta^{2}} \right)^{T/V} dz \int_{B}^{\infty} e^{-\frac{\theta}{\sigma^{2}} z^{2}} dx
\]

by reversing the order of integration.

We will in principle consider positive risk-aversion in option markets, characterised by \( \theta < 0 \). The second integral may be easily computed by the Barrow’s theorem:

\[
I(1/V, \theta, T; B) = \frac{1}{2\pi} \left( \frac{1}{\sigma^{2} / 2} \right)^{T/V} \int_{-\infty}^{\infty} \left( \frac{1}{z^{2} + \beta^{2}} \right)^{T/V} dz \int_{B}^{\infty} e^{-\frac{\theta}{\sigma^{2}} z^{2}} dx
= \frac{1}{2\pi i} \left( \frac{1}{\sigma^{2} / 2} \right)^{T/V} e^{\frac{\theta}{\sigma^{2}}} \int_{-\infty}^{\infty} e^{-iz B} \int_{-\infty}^{\infty} (z + i \frac{\theta}{\sigma^{2}})(z^{2} + \beta^{2})^{-T/V} dz
\] \hspace{1cm} [10a]

And as a real valued integral:

\[
I(1/V, T, \theta, B) = \frac{1}{2\pi} \left( \frac{1}{\sigma^{2} / 2} \right)^{T/V} e^{\frac{\theta}{\sigma^{2}}} \int_{-\infty}^{\infty} \frac{-\theta}{\sigma^{2}} \cos Bz - zs Bz}{(z^{2} + \left( \frac{\theta}{\sigma^{2}} \right)^{2})(z^{2} + \beta^{2})^{-T/V}} dz
\] \hspace{1cm} [10b]

\(^{14}\) This requires \( \frac{1}{V} - \frac{\alpha^{2}}{2} > 0 \), but as it was pointed out in the previous section, this is a restriction of the model parameters. The cases studied in Carr-Geman-Madan-Yor ’99 are all very far from violating this condition.
As \( \theta + \sigma^2 \) might be positive, the integrals in \([9a]\) may have different parameter restrictions. We will calculate \( I \) (even for the relevant case where \( \theta < 0 \)) considering that \( \theta \) can be positive. To take into account this fact, we will use that

\[
\int_{-\infty}^{\infty} e^{f(x)} \, dx = e^{-\theta} \left( \int_{-\infty}^{\infty} e^{f(x)} \, dx + I \left( 1 - \theta - \frac{\sigma^2}{2}, \theta + \sigma^2; T; B \right) \right)
\]

and the call price will be, if \( \theta < 0 \):

\[
C_T = S_0 \left( I \left( 1 - \theta - \frac{\sigma^2}{2}, \theta + \sigma^2; T; B \right) \right) - K e^{-rT} I \left( 1, \theta, T; B \right) \quad \text{[9b]}
\]

being \( I \) if \( a > 0 \) and zero otherwise.

If \( \theta + \sigma^2 = 0 \), we can obtain a closed form using Gradshteyn-Ryzhik’65 ([6.561.4]).

If we accept negative risk-aversion, the second term will include a function like \( I \) too.

\([10\ a,b]\) can be computed by means of the Fast Fourier Transform (FFT). This is not possible expressing the call price by Fourier-inversion of \([3]\), a method generalised in Bakshi-Madan’98. Given the great computational advance that the FFT implies, Carr-Madan’98 propose an alternative way to perform it. It consists of introducing an external factor, making possible direct application of FFT.

Obviously, the solution must be independent of the damping factor, so the method carries a computational cost. Analytically, \([10a]\) is simpler than its two former alternatives. Moreover, the call price can be expressed as combination of real valued integrals \([10b]\).

### 3.- PRICING OPTIONS BY INTERPOLATION

The following procedure will enable us to compute option prices even more efficiently than using the FFT. With such purpose in mind, we will enquire about the complex nature of \([10a]\). As it was established, the integrand of \([10a]\) is an analytic function in \( \mathbb{C} \) (excepting at some points) if time to maturity is an integer multiple of the kurtosis parameter \( \nu \). For those maturity dates, the integrand is analytical excepting at three \(^{15}\) pure imaginary values. Two of them are symmetric located respect the origin and they are poles of order \( T/\nu \). Between them there exists a simple pole at \( -i\theta /\sigma^2 \). This expression is easily computable using the Residue theorem. It becomes (but constants) the sum of the residues of \([10a]\) at the poles situated above the real axis if \( B<0 \) and below otherwise.

Thus, the probability elements in option prices turn out to be elementary functions, valid for maturities \( T= \nu, 2\nu, 3\nu, \ldots \). The option prices increase with \( T \), being its dependence specially smooth. This makes possible to obtain option prices by interpolation.

Pricing options by interpolation is not new in the finance literature. Schroder’89 approximates the Constant Elasticity Variance model interpolating normal distributions. We need not any special function.

The smoothness of the price with respect \( T \) guarantees a rapid convergence to its true value, so few values are required. Obviously, these will be chosen around the contract maturity date.

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\(^{15}\) Four, if considering the real valued expression \([10b]\).
An alternative is to perform the interpolation by letting to vary the parameter \( v \) instead of \( T \). Taking \( v = T, \frac{T}{2}, \frac{T}{3}, \ldots \) improves the results, because prices are less sensitive to changes in \( v \) than in \( T \). It will also be used when analysing the American put.

So instead of evaluating \([10a]\) in hundreds of points in order to perform the FFT, we may calculate the probability elements corresponding to several values of \( v \). Then, probabilities (and prices) are given by elementary functions.

Denoting \( T/v \) as \( n \), the integrand of \([10a]\) as \( f(z) \) and making use of standard results of Complex Variable Theory, we have for \( B < 0 \):

\[
I(\frac{1}{v}, \theta; T; B) = \frac{1}{2\pi i} \left( \frac{1}{\sigma^2 v / 2} \right)^n e^{\frac{\theta}{\sigma^2} B} \int_{-\infty}^{\infty} f(B, z) dz = \frac{1}{2\pi i} \left( \frac{1}{\sigma^2 v / 2} \right)^n e^{\frac{\theta}{\sigma^2} B} 2\pi i \sum_{k} \text{Re} \ s_{z=k} f(B, z)
\]

\[
= \left( \frac{1}{\sigma^2 v / 2} \right)^n e^{\frac{\theta}{\sigma^2} B} \left( \text{Re} \ s_{z=-\frac{n}{\sigma^2}} f(z) + \text{Re} \ s_{z=\frac{n}{\sigma^2}} f(B, z) \right)
\]

\[
= \left( \frac{1}{\sigma^2 v / 2} \right)^n e^{\frac{\theta}{\sigma^2} B} \lim_{n \to \infty} \left[ (z + i \frac{n}{\sigma^2}) f(B, z) \right] + \frac{d^n}{dz^n} \left[ (z - i\beta)^n f(B, z) \right]_{z=+i\beta}
\]

\[
= 1 + \left( \frac{1}{\sigma^2 v / 2} \right)^n e^{\frac{\theta}{\sigma^2} B} \left( \frac{d^n}{dz^n} \left[ (z - i\beta)^n f(B, z) \right]_{z=+i\beta} \right)
\]

For \( B > 0 \), the exponential in \( f(z) \) would explode, and the path of integration had to be chosen below the real axis.

As examples, we will make explicit the cases \( T/v = 1, 2 \):

\[
I(\frac{1}{v}, \theta, 1; B) = 1 - \left( \frac{1}{\sigma^2 v} \right) e^{\frac{\theta}{\sigma^2} (\beta + B)}
\]

\[
I(\frac{1}{v}, \theta, 2; B) = 1 - \left( \frac{1}{\sigma^2 v} \right)^2 e^{\frac{\theta}{\sigma^2} (\beta + B)} \frac{\left( 2\beta + \frac{n}{\sigma^2} \right) B}{\left( \frac{n}{\sigma^2} + \beta \right)^2 \beta^3}
\]

if \( B < 0 \) and

\[
I(\frac{1}{v}, \theta, v; B) = \left( \frac{1}{\sigma^2 v} \right) e^{\frac{\theta}{\sigma^2} (\beta - \beta)} (-\beta)
\]

\[
I(\frac{1}{v}, \theta, 2v; B) = \left( \frac{1}{\sigma^2 v} \right)^2 e^{\frac{\theta}{\sigma^2} (\beta - \beta)} \frac{\left( -2\beta + \frac{n}{\sigma^2} \right) B}{\left( \frac{n}{\sigma^2} - \beta \right)^2 (-\beta^3)}
\]

if \( B > 0 \).

Maturities below \( v \) represent an inconvenient, because interpolation cannot be applied. Extrapolation is feasible but is less accurate than interpolating and as the option maturity decreases (relative to \( v \)) the valuation error increases.
This method can also be applied to American option pricing. Carr’98 develops a method for pricing American options in a BS context. It consists of dividing $T$ in $N$ periods of equal length. Then, he approximates the *gaussian* returns density by a *gaussian* conditional on $T$ given by an *Erlang* distribution. As $N \to \infty$, the *Erlang* distribution approach the Dirac delta. So in the limit, it replicates the *normal* distribution corresponding to the BS case. This may be easily performed since the valuation of an American put conditional on an *Erlang* distributed $T$ admits simple expressions. As it converges relatively fast\(^{16}\), the method provides a good performance. But the intermediate values converging to the BS price may be viewed as the prices of options with $v = T, T/2, T/3,...$ if employing for pricing the VG model instead of the BS framework. So interpolating prices of options with such values of $v$ we can approach American option prices. Contrary to what could be expected, pricing American puts in a BS framework is more expensive than using the VG model.

4.- GENERALIZATION

In this section we will generalise the procedure proposed in Section 2. We will consider a wide class of price processes useful in finance, representing an important subclass of the processes with completely monotone Lévy measures. It is obtained from the gamma case by convolution and it may alternatively be characterised through the behaviour of the corresponding Lévy density. Examples are provided by the *CGMY* model or the $a$-stable processes with $a<1$\(^{17}\).

Let us introduce the definitions in terms of Lévy densities. Then, we will present a generalisation of the integral VG pricing formula.

**Definition:** A Lévy measure $k(x)$ is in the class of generalised gamma convolution if the size weighted Lévy density $x k(x)$ is completely monotone.

A process $X_t$ is in the class of the generalised gamma convolution if its size weighted Lévy density is completely monotone.

Obviously, the processes lying in the generalised gamma convolution have completely monotone Lévy measures.

The gamma process is the building block for processes in the gamma convolution as exponential Lévy densities are the building blocks of completely monotone densities\(^{18}\). Now we present a characterisation of the generalised gamma convolution densities.

**Proposition 1:** Let $X_t$ be a process in the class of the generalised gamma convolution. Then, its density function is a mixture of gamma processes. Moreover, its characteristic function has the form

$$
\phi_{X_t} = \exp \left\{ \int_{R} \log \left( \frac{e^{-x}}{x} \right) K(dc) \right\}, \quad \text{with } K(dc) \text{ the non-negative mixing measure, requiring }
\int_{0}^{\infty} \left| \log(c) \right| K(dc) < \infty, \int c K(dc) < \infty
$$

\(^{16}\)Specially using Richardson extrapolation.

\(^{17}\)Carr-Geman-Madan-Yor’00 and McCulloch’78, respectively.

\(^{18}\)They can be obtained by calculating the Laplace transform of absolute continuous densities.
Proof: Geman-Madan-Yor’98, appealing to Bondesson’92.

If the weighting mixture function is a Dirac delta, then the process is a gamma one.

Observation: A VG process is a mixture (a difference) of gamma processes, so they are in the generalised gamma convolution.

The next result links the random time and the logprice process.

Proposition 2: A symmetric process $X(t)$ in the generalised gamma convolution can be expressed as a BM evaluated at a random time $T(t)$ belonging to the generalised gamma convolution. Furthermore, if the mixing measure $K(dc)$ corresponding to $X(t)$ is expressible as $g(c)dc$, their characteristic functions are:

$$
\phi_{T(t)}(u) = \exp \left\{ \int \log \left( \frac{e^{itc}}{\sqrt{2\pi c}} \right) g(c) dc \right\}, \quad \phi_{W(T(t))}(u) = \exp \left\{ \int \log \left( \frac{e^{itc}}{\sqrt{2\pi c}} \right) g(c) dc \right\}
$$

The proof is in Geman-Madan-Yor’98. It must be noted that $g(c)$ inherits properties from the VG restriction $c - \theta - \frac{\sigma^2}{2} > 0$.

Now we might generalise the pricing formula of Section 2 to processes belonging to the generalised gamma convolution.

Proposition 3: Let $X_i = \theta \tau(t) + \sigma W_{\tau(t)}$ be a logprice process with $\tau(t)$ in the generalised gamma convolution (equivalently, $X_i$ is in the gamma convolution). The call price is then given by an expression of the form:

$$
C_i = S_0 J(\overline{\beta}, -\theta - \sigma^2 / 2, \theta + \sigma^2, \sigma, T; B) - Ke^{-rT} J(\overline{\beta}, 0, \theta, \sigma, T; B)
$$

where: $J(\overline{\beta}, 0, \theta, T; B) = \frac{1}{2\pi} e^{\frac{\theta^2}{\sigma^2}} \int_{-\infty}^{\infty} e^{-iuB} \frac{\phi_{X_i T_i}(u; \overline{\beta})}{u + i\frac{\theta}{\sigma}} du$  \[11\]

$\phi_{X_i T_i}(u; \overline{\beta})$ is a real valued function and

$\overline{\beta}$ is the set of parameters of $K(dc)$ (the mixing measure of $\tau(t)$)

Proof: Both probabilities elements are in the same parametric class because the processes in the generalised gamma convolution are mixtures of VG processes. The expressions for such probability elements are obtained making the change of variable $z = u - i\theta / \sigma^2$ and then applying Cauchy’s Theorem.

The characteristic function of $X(t) = \theta \tau(t) + \sigma W_{\tau(t)}$ evaluated at $z + i\frac{\theta}{\sigma^2}$ is

$$
\phi_{X_i}(z + i\frac{\theta}{\sigma^2}) = \int_{-\infty}^{\infty} \exp \left( -\frac{z^2}{2\sigma^2} + i\frac{\theta z}{\sigma^2} \right) f_i(s) ds \equiv \phi_{X_i}(z), \text{ a real valued expression}
$$
When the characteristic function of \( \tau(t) \) is known, [11] gives us the pricing formula. It may happen that the characteristic function of \( \tau(t) \) were unknown, being known the characteristic function of \( X_t \). Then, the precedent proposition ensures the existence of a pricing formula in the fashion of [11], but its particular form has to be found. The CMY model represents an example of the first situation as the CGMY model of the second one. We will now restrict our attention to both models.

The **CGMY model**: it represents a generalisation of the VG model and is intended to capture the fine structure of assets returns. This model is proposed in Carr-Geman-Madan-Yor'00 and the price processes may be characterised by its Lévy density:

\[
\begin{align*}
  k_{CGMY} &= \begin{cases} 
    Ce^{-|x|} & x < 0 \\
    \frac{C}{|x|^{1+Y}} & x > 0
  \end{cases} 
  C,G,M>0, \ 0<Y<1 \ or \ 1<Y<2 \quad [12]
\end{align*}
\]

If \( Y=0 \), the VG model is recovered. If \( Y>1 \), the corresponding process is of infinite variation. The positive risk aversion corresponds to \( G<M \) and we will restrict our attention to that case. In Carr-Geman-Madan-Yor'00 that condition holds in all cases under study. The characteristic function may be found in such paper and is given by:

\[
\phi_{CGMY}(u) = \exp \left\{ iuCT(-Y)((M - iu)^Y - M^Y + (G + iu)^Y - G^Y) \right\} \quad [13]
\]

In a similar way to the VG case, we obtain:

\[
C_I = S_0 e^{\alpha T} \int_B e^{\gamma x} f_X(x) dx - Ke^{-\gamma T} \int_B f_X(x) dx =
\]

\[
= S_0 e^{\alpha T} \int_B e^{\gamma x} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \exp \left\{ iuCT(-Y)((M - iu)^Y - M^Y + (G + iu)^Y - G^Y) \right\} du -
\]

\[
- Ke^{-\gamma T} \int_B dx \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \exp \left\{ iuCT(-Y)((M - iu)^Y - M^Y + (G + iu)^Y - G^Y) \right\} du =
\]

\[
= S_0 e^{\alpha T} \hat{I}(C,G,M,Y;B) - Ke^{-\gamma T} \hat{I}(C,G,M,Y;B)
\]

As we know after Proposition 3, both integrals belong to the same parametric class. In particular:

\[
\hat{I} = \int_B e^{\gamma x} dx \int_{-\infty}^{\infty} e^{-iux} \exp \left\{ iuCT(-Y)((M - iu)^Y - M^Y + (G + iu)^Y - G^Y) \right\} du =
\]

\[
= \int_B dx \int_{i(-1)}^{i(1)} e^{\gamma x} dx \int_{-\infty}^{\infty} e^{-iux} \exp \left\{ iuCT(-Y)((M - iu)^Y - M^Y + (G + iu)^Y - G^Y) \right\} du
\]

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Making the change $-iu + l = -iz$:

$$\hat{I} = \int_{B}^{\infty} dx \int_{-\infty}^{\infty} e^{-iz} \exp \left\{ TCT(-Y) \left[ (M - 1 - iz)^Y - M^Y + (G + 1 + iz)^Y - G^Y \right] \right\} dz.$$  

Like in the VG case, the model parameters have numerical restrictions imposed by the convexity correction. In fact, they come from the VG restriction $\frac{1}{v} - \theta - \frac{\sigma^2}{2} > 0$, inherited by $K(dc)$ and so by $\phi_{CGMY}(u)$.

In the present case the convexity correction is given from:

$$e^{-at} = \phi_{CGMY}(-i) = \exp \left\{ TCT(-Y) \left[ (M - 1)^Y - M^Y + (G + 1)^Y - G^Y \right] \right\}.$$  

Since $Y$ is not an integer and $o$ must be a real number, $M$ cannot be smaller than one. This restriction makes the integrand analytical in a certain region, as it occurred in the VG case. We may write:

$$\hat{I} = \int_{B}^{\infty} dx \int_{-\infty}^{\infty} e^{-iz} \exp \left\{ TCT(-Y) \left[ (M - 1 - iz)^Y - M^Y + (G + 1 + iz)^Y - G^Y \right] \right\} dz$$  

As $M > 1$, the new parameters $C, G + 1, M - 1, Y$ keep verifying the initial restrictions in [12]. It permits to write $\hat{I} = I(C, G + 1, M - 1, Y; B)$ and $C_T = S_0 I(C, G + 1, M - 1, Y; B) - Ke^{-\gamma T} I(C, G, M, Y; B)$.

In the Appendix it is shown that

$$I(C, G, M, Y, B) = \frac{1}{2\pi} e^{-B(M-G)/2} \int_{-\infty}^{\infty} e^{-ivB} \exp \left\{ TCT(-Y) \left[ (M + G) - (M + G + iv)^Y + (M + G) - 2(M + G)^y \right] \right\} dv \quad [14]$$  

This expression may be directly calculated by the FFT, and a dampening factor is again unnecessary. Equivalently, it can be written as:

$$I(C, G, M, Y; B) = \frac{1}{2\pi} e^{-M-GB} \int_{-\infty}^{\infty} \exp \left\{ TCT(-Y) \left[ (M + G - iv)^Y + (M + G + iv)^Y - 2(M + G)^Y \right] \right\} dv$$  

$$= \frac{1}{2\pi} e^{-M-GB} \int_{-\infty}^{\infty} \exp \left\{ TCT(-Y) \left[ (M + G - iv)^Y + (M + G + iv)^Y - 2(M + G)^Y \right] \right\} dv$$  

We can go further by expressing these integrals as real forms. The symmetric $CGMY$ process implicit in [14] is a particular case of the truncated Lévy flights studied in Koponen’95. The characteristic function there given is:

$$\phi_{CGMY}(u) = \exp \left[ TC_1 C_2 - TC_2 \left( \frac{u^2 + (M+G)^2}{\cos(\frac{\pi Y}{2})} \right)^{Y/2} \cos \left( \frac{\text{Yarctan} \left( \frac{|u|}{M+G} \right)}{\frac{\pi Y}{2}} \right) \right]$$  

and $C_1 = \frac{(M + G)^Y}{\cos(\pi Y / 2)}, \quad C_2 = \frac{2C \pi \cos(\pi Y / 2)}{\Gamma(Y) \sin(\pi Y)}$.
The **CMY** model: It consists of a BM evaluated at a random time given by a **CGMY** process where \( G=0 \). Obviously, the time process is non-decreasing and one-sided. In contrast to the VG model, the difference of two of these random times is not a **CGMY** process. This can easily be checked looking at their characteristic functions.

The characteristic function of the **CMY** model can be found in Geman-Madan-Yor’99:

\[
\phi_{CMY}(u) = e^{-TCY}\left[1 - (M - \theta + u^2/2)^V - M^Y\right]
\]

In order to obtain a pricing formula we follow similar steps to precedent cases. First of all, we demonstrate that both integrals are in the same parametric class. The call price may then be expressed as:

\[
C_T = S_0 I(\theta + \sigma^2, M - \theta - \sigma^2/2, Y; B) - Ke^{-rT}I(\theta, M, Y; B)
\]

For this reduction we used \( M - \theta - \sigma^2/2 > 0 \). This is necessary because \( M>0 \), and \( M - \theta - \sigma^2/2 \) plays the role of \( M \) in the first term. Moreover, we need such inequality to apply Cauchy’s theorem and substitute the complex axis of integration by the real one.

We may proceed like that since \( M - \theta - \sigma^2/2 > 0 \), because \( \phi_{CMY}(-i) \) must be real.

Finally, \( I(\cdot) \) can be written as:

\[
I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ivB}}{iV - \frac{\theta + \sigma^2}{\alpha^2}} \exp\left\{ -TCY(M - i\upsilon \theta + \sigma^2 \upsilon^2/2)^V - M^Y \right\}d\upsilon
\]

also expressible in real terms.

**Exponential price shocks:** We now consider a process that represents the building block of the completely monotone processes and which does not belong to the gamma convolution. We will see that the proposed method is not valid for such process, because the terms \( P_1, P_2 \) are not in the same parametric class. It is given by a **Poisson** process with exponential prevailing price order size, instead of **gaussian** (Madan-Geman-Yor’98). Its Lévy density is given by

\[
k_e(x)dx = e^{-a_1x}dx.
\]

This also can be written as \( X_t = \theta T(t) + \sigma W_{T(t)} \). The characteristic functions of \( X_t \) and \( T(t) \) are respectively:

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19 It can also be computed by conditioning the **Brownian** characteristic function on the random time, **CMY** distributed. Its density may be expressed Fourier-inverting its characteristic function. The resultant double integral is then reduced to a single one by reversing the order of integration and carrying out the second integral. The remaining integral may be very easily calculated by applying the Residue theorem.
\[ \phi_{\log S=X_i}(u) = \exp \left\{ \frac{2it}{a} \left( \frac{a^2}{a^2 + \gamma^2 (u^2 - 2i\theta/\sigma^2)} - 1 \right) \right\} \]

\[ \phi_{T(t)}(u) = \exp \left\{ \frac{2it}{a} \left( \frac{a^2}{a^2 - iiu} - 1 \right) \right\} \]. So

\[ \int_{-\infty}^{\infty} e^{iu\phi_t(x)} dx = \int_{-\infty}^{\infty} dx e^{\frac{t}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2\sigma^2}} f(x)dx = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} e^{\frac{(x-\theta)^2}{2\sigma^2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\tau \phi_t(u)}d\tau = 

= \int_{-\infty}^{\infty} dx \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-i(x+\sigma^2/2\tau)}{2\sigma^2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\tau(\frac{a^2}{a^2 - iiu})}d\tau 

= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{i(x+\sigma^2/2\tau)}{2\sigma^2}} \int_{-\infty}^{\infty} e^{-i\tau(\frac{a^2}{a^2 - iiu})}d\tau 

Making \(-iu + \theta + \sigma^2/2 = -i\nu\), we observe that the resultant expression \(e^{2it(\frac{a^2}{a^2 - \sigma^2/2 - iv})}\) is not in the same parametric class as \(e^{2it(\frac{a^2}{a^2 - iv})}\).

The same conclusion holds for a logprice driven by \(X_i = \theta N(t) + \sigma W_{N(t)}\), where \(N(t)\) represents a Poisson process. This case corresponds to a gaussian Lévy density, which is not completely monotone, and it is related to the Merton’76 model. As it was pointed out in Geman-Madan-Yor’98, the time change \(N(t)\) is akin to the number of trades, independently of the magnitude of them\(^{20}\). This is the time change observed to be relevant by Ané-Geman’00 in their empirical study of high frequency returns on the FTSE100 futures index.

Although the last two examples have probability elements of different parametric classes, these probability terms can be computed by FFT using the method given above.

5.- CONCLUSIONS

A method of inverting the distribution transform in order to price options is proposed. It is based on complex variable theory and it permits to obtain prices for a wide class of processes as simple integrals. This offers numerous advantages.

Firstly, such integrals can be directly computed invoking the Fast Fourier Transform. Moreover, they can be expressed as real valued integrals. It means an advance from previous works. In such former

\(^{20}\) Empirical support is provided by Jones-Kaul-Lipson’94
formulations and in order to apply the FFT, it was introduced an external factor which complicated unnecessarily the expressions.
Restricting ourselves to the VG model, another interesting feature is that option prices can be calculated by interpolation, because the probability elements admit elementary functional expressions for some maturity dates. This can also be applied to American put pricing, employing the results of Carr’98.
The method is also generalised to the wide class of processes in the generalised gamma convolution, obtaining similar expressions to the VG case.
Finally, relevant models in option pricing like the CGMY or the CMY are considered.

Appendix

CGMY model call price:

Making the change \( iu = iv + \frac{M-G}{2} \),

\[
I(C, G, M, Y; B) = \int_B^\infty dx \frac{1}{2\pi} \int_{-\infty}^\infty e^{-iu\tau} \exp\left\{ T\tau (Y) \left[ \left( M - iv \right)^Y - M^Y + \left( G + iv \right)^Y - G^Y \right] \right\} du =
\]

\[
\int_B^\infty dx \frac{1}{2\pi} \int_{-\infty}^{\infty+i(M-G)/2} e^{-(M-G)/2+iv\tau} \exp\left\{ T\tau (Y) \left[ \left( \frac{M+G}{2} - iv \right)^Y - M^Y + \left( \frac{M+G}{2} + iv \right)^Y - G^Y \right] \right\} d\tau =
\]

\[
\int_B^\infty e^{-(M-G)/2} dx \int_{-\infty+i(M-G)/2}^{\infty+i(M-G)/2} e^{-iv\tau} \exp\left\{ T\tau (Y) \left[ \left( \frac{M+G}{2} - iv \right)^Y - M^Y + \left( \frac{M+G}{2} + iv \right)^Y - G^Y \right] \right\} d\tau
\]

The integrand is analytic between the path of integration and the real axis. This justifies to write:

\[
I(C, G, M, Y, B) = \frac{1}{2\pi} \int_B^\infty e^{-\xi(M-G)/2} dx \int_{-\infty}^\infty e^{-iv\xi} \exp\left\{ T\xi (Y) \left[ \left( \frac{M+G}{2} - iv \right)^Y - M^Y + \left( \frac{M+G}{2} + iv \right)^Y - G^Y \right] \right\} d\xi =
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^\infty e^{-\mu(M-G)/2} \exp\left\{ T\mu (Y) \left[ \left( \frac{M+G}{2} - iv \right)^Y + \left( \frac{M+G}{2} + iv \right)^Y - 2\left( \frac{M+G}{2} \right)^Y \right] \right\} d\mu =
\]

\[
= \frac{1}{2\pi} e^{-B(M-G)/2} \int_{-\infty}^\infty e^{-Bv} \exp\left\{ T\tau (Y) \left[ \left( \frac{M+G}{2} - iv \right)^Y + \left( \frac{M+G}{2} + iv \right)^Y - 2\left( \frac{M+G}{2} \right)^Y \right] \right\} dv
\]
References


