Market Value of Life Insurance Contracts and Barrier Derivatives under Stochastic Interest Rates and Default Risk

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Outline of the Talk

1. Bibliography
2. Description of the Contracts
3. Asset and Interest Rate Model
4. Evaluation Method
5. Numerical Analysis of LICs
6. The Case of Exotic Options
A Short Actuarial Bibliography...

Brennan and Schwartz [1976]
Briys and de Varenne [1993, 1997]
Grosen and Jørgensen [1997, 2000, 2002]
Tanskanen and Lukkarinen [2003]
Jørgensen [2004]
...Complemented by some Major References from Finance

( Fortet [1943] )

Merton [1974]

Hull and White [1987]

Heath, Jarrow and Morton [1992]

Longstaff and Schwartz [1995]

Collin-Dufresne and Goldstein [2001]
## Capital Structure of the Insurance Company

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
</table>
| $A_0$  | $E_0 = (1 - \alpha)A_0$  
|        | $L_0 = \alpha A_0$      |

- $E_0 = \text{initial equity value}$
- $L_0 = \text{initial investment of the policyholders who all possess the same contract.}$
Simplified Description of the Contract

The policyholders’ investment $L_0$ yields the minimum guaranteed rate $r_g$ at contract expiry $T$.

$$L_T^g = L_0 e^{r_g T}$$

- **In case of No-default**: $A_T \geq L_T^g$
  
  Policyholders receive the guaranteed amount at $T: L_T^g$

- **In case of Default**: $A_T < L_T^g$
  
  Policyholders receive $A_T$. Equityholders receive nothing.
A Participating Policy

Policyholders are given a contractual part $\delta$ of the benefits of the company when the assets are sufficiently high at maturity:

$$A_T > \frac{L^g_T}{\alpha} \quad \text{where} \quad \alpha < 1.$$ 

Assuming no prior bankruptcy, policyholders receive at $T$:

$$\Theta_L(T') = \begin{cases} 
A_T & \text{if } A_T < \frac{L^g_T}{\alpha} \\
L^g_T & \text{if } \frac{L^g_T}{\alpha} \leq A_T \leq \frac{L^g_T}{\alpha} \\
L^g_T + \delta(\alpha A_T - L^g_T) & \text{if } A_T > \frac{L^g_T}{\alpha}
\end{cases}$$
The firm pursues its activities until $T$ if:

$$\forall t \in [0, T[ , \quad A_t > L_0 e^{rt} \triangleq B_t$$

Let $\tau$ be the default time

$$\tau = \inf\{t \in [0, T] \mid A_t < B_t\}$$

In case of prior insolvency, policyholders receive:

$$\Theta_L(\tau) = L_0 e^{rt\tau}$$
In this setting, the price of a participating contract writes as:

\[ V_L(0) = \mathbb{E}_Q \left[ e^{-\int_0^T r_s \, ds} \left( L_T^g + \delta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+ \right) \mathbbm{1}_{\tau \geq T} + e^{-\int_0^\tau r_s \, ds} L_0 e^{r_s \tau} \mathbbm{1}_{\tau < T} \right] \]

This contract can be split up into four simpler subcontracts:

\[ V_L = \widehat{GF} + \widehat{BO} - \widehat{PO} + \widehat{LR} \]

- \( \widehat{GF} \): the final guarantee.
- \( \widehat{BO} \): the "bonus option" which is the participating clause.
- \( \widehat{PO} \): the default put on which policyholders are short.
- \( \widehat{LR} \): the rebate paid in case of early default.
Interest Rate Modeling

The dynamics under $Q$ of the interest rate $r$ and the zero-coupon bonds $P(t,T)$ are:

$$dr_t = a(\theta - r_t)dt + \nu dZ^Q_1(t)$$

and:

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \sigma_P(t,T) dZ^Q_1(t)$$

We assume an exponential volatility for the zero-coupons:

$$\sigma_P(t,T) = \frac{\nu}{a} \left(1 - e^{-a(T-t)}\right)$$
The asset dynamics under the risk-neutral probability $Q$ are:

$$\frac{dA_t}{A_t} = r_t dt + \sigma dZ^Q(t)$$

where $Z^Q$ and $Z_1^Q$ are correlated $Q$-Brownian motions.

$$(dZ^Q.dZ_1^Q = \rho dt).$$

We decorrelate the interest rate and asset risks. Let $Z_2^Q$ be independent from $Z_1^Q$, then:

$$dZ^Q(t) = \rho dZ_1^Q(t) + \sqrt{1 - \rho^2} dZ_2^Q(t)$$
Let $Q_T$ be the $T$-forward-neutral measure. From Girsanov’s theorem, $Z_1^{QT}$ and $Z_2^{QT}$ are independent $Q_T$-Brownian motions when defined by:

$$dZ_1^{QT} = dZ_1^Q + \sigma_P(t, T)dt, \quad dZ_2^{QT} = dZ_2^Q$$

Under $Q_T$, the prices of $P(t, T)$ and $A_t$ follow the stochastic differential equations:

$$\frac{dP(t, T)}{P(t, T)} = (r_t + \sigma_P^2(t, T))dt - \sigma_P(t, T)dZ_1^{QT}$$

and

$$\frac{dA_t}{A_t} = (r_t - \sigma\rho\sigma_P(t, T))dt + \sigma \left( \rho dZ_1^{QT} + \sqrt{1 - \rho^2} dZ_2^{QT} \right)$$
Contract Valuation at $t = 0$

The contract's price writes in the Forward-Neutral universe:

$$V_L(0) = P(0, T) \left[ L_T^g (1 - E_1) ight.$$  

$$+ \, \alpha \delta (E_7 - E_2) - \delta L_T^g (E_8 - E_3)$$  

$$- \, L_T^g (E_9 - E_4) + E_{10} - E_5 + L_0 \, E_6 \right]$$
with the following quantities that remain to be computed:

\[
\begin{align*}
E_1 &= \mathbb{Q}_T[\tau < T] \\
E_2 &= \mathbb{E}_\mathbb{Q}_T \left[ A_T \mathbbm{1}_{A_T > \frac{L^g_T}{\alpha}, \tau < T} \right] \\
E_3 &= \mathbb{Q}_T \left[ A_T > \frac{L^g_T}{\alpha}, \tau < T \right] \\
E_4 &= \mathbb{Q}_T \left[ A_T < L^g_T, \tau < T \right] \\
E_5 &= \mathbb{E}_\mathbb{Q}_T \left[ A_T \mathbbm{1}_{A_T < L^g_T} \mathbbm{1}_{\tau < T} \right] \\
E_6 &= \mathbb{E}_\mathbb{Q}_T \left[ e^{\int_\tau^T \tau_s ds} e^{rg\tau} \mathbbm{1}_{\tau < T} \right] \\
E_7 &= \mathbb{E}_\mathbb{Q}_T \left[ A_T \mathbbm{1}_{A_T > \frac{L^g_T}{\alpha}} \right] \\
E_8 &= \mathbb{Q}_T \left[ A_T > \frac{L^g_T}{\alpha} \right] \\
E_9 &= \mathbb{Q}_T[ A_T < L^g_T] \\
E_{10} &= \mathbb{E}_\mathbb{Q}_T \left[ A_T \mathbbm{1}_{A_T < L^g_T} \right]
\end{align*}
\]
Methodology:  
Extended Fortet’s Approximation

Problem:  We need to know the law of $\tau$, first passage time of the assets beyond the default-triggering barrier.

► Longstaff and Schwartz (1995) use Fortet's results to approximate the density of $\tau$ in a problem similar to ours.

► Collin-Dufresne and Goldstein (2001) give a correction to the previous method.
Methodology:
Extended Fortet’s Approximation


- Here we go beyond their approaches by pricing general barrier derivatives.
First Passage Time Approximate Density

Let us recall the definition of $\tau$:

$$
\tau = \inf\{t \in [0, T] \mid A_t < L_0 e^{r g t}\}
$$

**Scheme’s Idea:** Approximate the density of $\tau$ at any time $t$ under $Q_T$ as a piecewise constant function.

- The interval $[0, T]$ is subdivided into $n_T$ subperiods:
  
  $$
t_0 = 0, \ldots, t_j = j \delta_t, \ldots, t_{n_T} = T
  $$

- The interest rate is discretized between $r_{\min}$ and $r_{\max}$ into $n_r$ intervals. $r_i = r_{\min} + i \delta_r$ are the discretized values of the interest rate.
The probability of the event \( \tau \in [t_j, t_{j+1}] \) with \( r \in [r_i, r_{i+1}] \) is denoted by :

\[ q(i, j) \]

Collin-Dufresne and Goldstein give a recursive formula to compute these probabilities, starting with :

\[ q(i, 1) = \Phi(r_i, t_1) \]

where one first computes \( q(i, 1) \) for each \( i \), and then \( q(i, j) \) recursively for \( j \geq 2 \) using :

\[
q(i, j) = \Phi(r_i, t_j) - \sum_{v=1}^{j-1} \sum_{u=0}^{n_r} q(u, v) \Psi(r_i, t_j | r_u, t_v)
\]

where \( \Phi \) and \( \Psi \) are completely known.
Expressions of $\Phi$ and $\Psi$

Let $\mathcal{N}$ be the cumulative function of the $Gauss(0, 1)$ law, then:

$$
\Phi(r_t, t) = f_r(r_t, t \mid l_0, r_0, 0) \mathcal{N}\left(\frac{h - \mu(r_t, l_0, r_0)}{\sqrt{\Sigma^2(r_t, l_0, r_0)}}\right)
$$

$$
\Psi(r_t, t \mid r_s, s) = f_r(r_t, t \mid l_s = h, r_s, s) \mathcal{N}\left(\frac{h - \mu(r_t, l_s = h, r_s)}{\sqrt{\Sigma^2(r_t, l_s = h, r_s)}}\right)
$$

where:

$$
f_r(r_t, t \mid l_s = h, r_s, s) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{(r_t - m)^2}{2v}}, \quad m = \mathbb{E}[r_t \mid r_s], \quad v = \text{Var}[r_t \mid r_s]
$$

and $l$ is defined by $l_t = \ln A_t - r_g t$. 

Empirical Density and Fortet’s Approximate Density

- Empirical Density
- \( n_r = 10 \) and \( n_T = 50 \)
- \( n_r = 50 \) and \( n_T = 200 \)

\[ \times 10^{-3} \]
Computation of the $E_i$ depending on $\tau$

Now, each expression $E_i$ can be computed easily even if it depends on $\tau$.

We detail how to valuate $E_1$ for instance; its exact expression is:

$$E_1 = Q_T[\tau < T]$$

We write $E_1$:

$$E_1 = \int_0^T ds \int_{-\infty}^{+\infty} dr_s \, g(r_s, s) \approx \sum_{j=1}^{n_r} \sum_{i=0}^{n_r} q(i, j)$$

where $g$ is the density of $(r_{\tau}, \tau)$. 

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Computation of the $E_i$ depending on $\tau$

Now, each $E_i$ can be computed easily even if it depends on $\tau$. We detail how to valuate $E_2$ for instance; its exact expression is:

$$E_2 = E_{Q_T} \left[ A_T 1_{\left\{ A_T > \frac{L^g_T}{\alpha}, \tau < T \right\}} \right]$$

Then, using conditional laws we obtain:

$$E_2 = e^{r_s T} \int_0^T ds \int_{-\infty}^{+\infty} dr_s g(r_s, s) E_{Q_T} \left[ e^{l_T} 1_{\left\{ l_T > \ln \left( \frac{L_0}{\alpha} \right) \right\}} \mid l_s = h, r_s, s, \tau = s \right]$$

As $\mathcal{L} (l_t | F_s, r_t) = Gauss \left( \mu, \Sigma^2 \right)$ and as we know the transition density of $r : f_r$ we can compute $E_2$ discretizing the integrals.
Let $X$ be a random variable with law $\mathcal{N}(m, \sigma^2)$, we denote
\[
\Phi_1(m; \sigma; a) = \mathbb{E}[e^X 1_{e^X > a}] = \exp \left( m + \frac{\sigma^2}{2} \right) \mathcal{N} \left( \frac{m + \sigma^2 - \ln(a)}{\sigma} \right)
\]

$E_2$ then admits the simpler expression:
\[
E_2 = e^{r_g T} \int_0^T ds \int_{-\infty}^{+\infty} dr_s g(r_s, s) \int_{-\infty}^{+\infty} dr_T f_T(r_T | r_s, s, l_s) \Phi_1 \left( \hat{\mu}_{s,T}; \hat{\Sigma}_{s,T}; \frac{L_0}{\alpha} \right)
\]

The extended Fortet’s approximation of $E_2$ writes:
\[
E_2 = e^{r_g T} \sum_{j=1}^{n_T} \sum_{i=0}^{n_r} \sum_{k=0}^{n_r} \delta_r f_T(r_k | r_i, t_j, l_t) \Phi_1 \left( \hat{\mu}_{t_j,T}; \hat{\Sigma}_{t_j,T}; \frac{L_0}{\alpha} \right) q(i, j)
\]
**Contract's Fair Value**

**Definition**: The initial investment of policyholders \( L_0 = \alpha A_0 \) must be equal to the contract market value at \( t = 0 \).

The Parameters:

- \( r_g \): minimum guaranteed interest rate
- \( \delta \): participating benefits

cannot be fixed arbitrarily: they obey regulatory constraints, and need to be set such as to make the contract fair between the insurer and the policyholder.
How to fix the Parameters of a Fair Contract?

We use a root search algorithm on the following equation to find the fair value of a parameter, *ceteris paribus*:

\[ L_0 = \{\text{Contract Value at } t = 0\} \]
Numerical Analysis

We set our parameter range according as:

<table>
<thead>
<tr>
<th>$A_0$</th>
<th>$a$</th>
<th>$\nu$</th>
<th>$\theta$</th>
<th>$r_0$</th>
<th>$\rho$</th>
<th>$\sigma$</th>
<th>$T$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.4</td>
<td>0.008</td>
<td>0.06</td>
<td>0.03</td>
<td>-0.02</td>
<td>0.1</td>
<td>10</td>
<td>0.7</td>
</tr>
</tbody>
</table>

$L_0 = \alpha A_0 = 70$

Contract Maturity: 10 years
Contract Value w.r.t. $\delta$

the participating coefficient

$r_g = 2.6\%$
Contract Value w.r.t. $r_g$
Minimum Guaranteed Rate
$\delta = 89.8\%$
Participating coefficient $\delta$

w.r.t. $r_g$ Minimum Guaranteed Rate

![Graph showing the relationship between $\delta$ and $r_g$ for different values of $\sigma$. The graph shows two curves: one for $\sigma=10\%$ (dashed line) and one for $\sigma=20\%$ (solid line). The x-axis represents $r_g$ ranging from 0.01 to 0.05, and the y-axis represents $\delta$ ranging from 0 to 1. The curves demonstrate how $\delta$ decreases as $r_g$ increases for both values of $\sigma$.](image-url)
## Numerical Results

<table>
<thead>
<tr>
<th>Method</th>
<th>GF</th>
<th>BO</th>
<th>PO</th>
<th>LR</th>
<th>Contract</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Extended Fortet</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n_T = 200, n_T = 50$</td>
<td>28.11</td>
<td>89.05</td>
<td>0.09</td>
<td>1.27</td>
<td>60.9967</td>
<td>2 min</td>
</tr>
<tr>
<td>$n_T = 500, n_T = 50$</td>
<td>28.11</td>
<td>89.03</td>
<td>0.09</td>
<td>1.29</td>
<td>69.9996</td>
<td>10 min</td>
</tr>
<tr>
<td><strong>Monte-Carlo</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$step = 1/12$</td>
<td>28.10</td>
<td>89.28</td>
<td>0.14</td>
<td>1.30</td>
<td>70.1108</td>
<td>15 min</td>
</tr>
<tr>
<td>$step = 1/52$</td>
<td>28.11</td>
<td>89.14</td>
<td>0.13</td>
<td>1.31</td>
<td>70.0451</td>
<td>1h20</td>
</tr>
<tr>
<td>$step = 1/365$</td>
<td>28.14</td>
<td>89.07</td>
<td>0.13</td>
<td>1.30</td>
<td>70.0201</td>
<td>1 day</td>
</tr>
</tbody>
</table>
Conclusion on LICs

- A study of relevance in the context of the new IAS and IFRS Standards
- A new method to price standard life insurance guarantees (guaranteed capital and minimum rate with participating bonuses when interest rates are stochastic and the possible default of the company is taken into account)
- The next step is to price supplementary options typical to life insurance contracts (surrender and conversion options, capital paid upon death and not at a fixed time)
Shark Options are Barrier Options with Rebate.

We consider the following example:

\[
C(t, T) = E_Q^t \left[ \exp \left( - \int_t^T r_s ds \right) \left( 1 + R_T \right) \mathbb{1}_{S_{\text{max}} \leq H} + \beta \mathbb{1}_{S_{\text{max}} > H} \right]
\]

where \( R_T = \frac{(S_T - S_0)^+}{S_0} \), the option is an \textit{up and out} one, and the rebate is \( \beta \).
Replace $A$, the assets of the insurance company, by $S$, the exotic option’s underlying; replace also a down and out default barrier by an up and out deactivating barrier... and nothing changes.
The Shark’s equilibrium price can therefore be expressed by:

\[ C(0, T) = P(0, T) (\beta E_1 + E_2 + E_3) \]

where:

\[ E_1 = \sum_{j=0}^{n_T} \sum_{i=0}^{n_r} q(i, j) \]

\[ E_2 = \mathcal{N} \left( \frac{l_0 - MT}{\sqrt{VT}} \right) - \sum_{j=0}^{n_T} \sum_{i=0}^{n_r} \sum_{k=0}^{n_r} \delta_{r_{fr}}(r_k | r_i, t_j, l_t_j) \mathcal{N} \left( \frac{l_0 - \hat{\mu}_{t_j,T}}{\sqrt{\Sigma_{t_j,T}}} \right) q(i, j) \]
Shark Options : Stochastic Barriers

When the barrier is proportional to a zero-coupon bond \((P(t, T)\) at time \(t\)), not only semi-closed but closed form formulae can be obtained by appealing to the Dubins-Schwatz theorem:

\[
E_1 = \mathcal{N} \left( \frac{\ln \left( \frac{S_0}{KP(0,T)} \right) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}} \right) + \frac{S_0}{KP(0,T)} \mathcal{N} \left( \frac{\ln \left( \frac{S_0}{KP(0,T)} \right) + \frac{\tau(T)}{2}}{\sqrt{\tau(T)}} \right)
\]

\[
E_2 = \mathcal{N} \left( \frac{\ln \left( P(0, T) \right) + \frac{\tau(T)}{2}}{\sqrt{\tau(T)}} \right) - \frac{S_0}{KP(0,T)} \mathcal{N} \left( \frac{\ln \left( \frac{S_0^2}{K^2P(0,T)} \right) + \frac{\tau(T)}{2}}{\sqrt{\tau(T)}} \right)
\]

...