An “Almost Exact” Simulation Method for the Heston Model

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The Heston Model

\[ dF_t = \sqrt{V_t} F_t \, dW_1 \]

\[ dV_t = \kappa_t (\theta_t - V_t) \, dt + \sigma_t \sqrt{V_t} \, dW_2 \]

\[ dW_1 dW_2 = \rho_t \, dt \]

- In Fourier Space, two dimensional Heston PDE becomes one dimensional

=> Vanilla prices have closed form solutions
The Heston Model (II)

• Pricing involves numerically integrating analytical solutions of Riccatti Equations.

• Time dependent case (with piecewise constant parameters) is no more complicated than the flat case!!!

• The Riccatti Equations must be solved in sections of time in which the parameters are constant, working backwards from maturity.

• Computational time \(\approx\) number of time sections * time to calculate flat Heston.
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An Exact Simulation Method

- Black Scholes has analytical solution for SDE => we can “long jump” in Monte Carlo

\[ S_t = S_0 e^{\left( r - \frac{1}{2} \sigma^2 \right) t + \sigma \sqrt{t} N(0,1) } \]

- In Stochastic Vol Models we have to do many little timesteps (SLOW!)

- Much research on better discretisation methods than Euler (e.g. Predictor-Corrector, Milstein…)

- Broadie and Kaya presented an “Exact Simulation Method” in their paper of 2003
An Exact Simulation Method (II)

Advantages of an exact simulation method are

• We need only simulate the dates of importance for the financial product we are modeling.

• The simulations have no bias in the error so we can construct valid Confidence Intervals.

• The Order of Convergence improves from $\frac{1}{3}$ to $\frac{1}{2}$.

• Greeks can be calculated using the same “tricks” as with Black-Scholes model because the forward price is lognormal conditional upon the path of the variance.
An Exact Simulation Method (III)

\[ F_t = F_u \exp\left[ -\frac{1}{2} \int_u^t V_s ds + \rho \int_u^t \sqrt{V_s} dW_s^{(1)} + \sqrt{1 - \rho^2} \int_u^t \sqrt{V_s} dW_s^{(2)} \right] \]

\[ V_t = V_u + \kappa \theta (t - u) - \kappa \int_u^t V_s ds + \sigma \int_u^t \sqrt{V_s} dW_s^{(1)} \]

\[ dW_s^{(1)} dW_s^{(2)} = 0 \]

- Generate a sample from the distribution of \( V_t \) given \( V_u \)
- Generate a sample from the distribution of \( \int_u^t V_s ds \) given \( V_t \) and \( V_u \)
- Recover \( \int_u^t \sqrt{V_s} dW_s^{(1)} \) from the above equation given \( V_t, V_u \) and \( \int_u^t V_s ds \)
- Generate a sample from the distribution of \( F_t \) given \( \int_u^t \sqrt{V_s} dW_s^{(1)} \) and \( \int_u^t V_s ds \)
1) Generate a sample from the distribution of $V_t$ given $V_u$

- The distribution is known and is a non-central chi-squared distribution

- Samples can be generated using combinations of Poisson, Gamma and Normal variates

- Sankaran (1963) shows that one can transform a non-central chi squared variate into an approximately normal variate. We use this to calculate confidence intervals for $V_t$ which we use in the next section.
An Exact Simulation Method (V)

2) Generate a sample from the distribution of given $V_t$ and $V_u$

- Brute force method: the CDF is given by a numerical integral of the known characteristic function of the distribution. We can generate samples from uniform deviates by inverting the CDF using Newton’s method.

- The characteristic function is computationally expensive as it involves modified Bessel functions of the first kind with a complex argument

- We should think about caching as much as we can!
An Exact Simulation Method (VI)

The characteristic function is given by

\[
\Phi(a; V_u, V_t) = \frac{\gamma(a)e^{-\frac{1}{2}(\gamma(a)-\kappa)(t-u)}(1-e^{-\kappa(t-u)})}{\kappa(1-e^{-\gamma(a)(t-u)})} \\
\times \exp\left\{ \frac{V_u + V_t}{\sigma^2} \left[ \frac{\kappa(1+e^{-\kappa(t-u)})}{1-e^{-\kappa(t-u)}} - \frac{\gamma(a)(1+e^{-\gamma(a)(t-u)})}{1-e^{-\gamma(a)(t-u)}} \right] \right\}
\]

Bessel functions of the first kind with complex argument

\[
\times \frac{I_{0.5d-\gamma}}{\sqrt{V_u V_t}} \left[ \frac{4\gamma(a)e^{-0.5\gamma(a)(t-u)}}{\sigma^2(1-e^{-\gamma(a)(t-u)})} \right]
\]

\[
\times \frac{I_{0.5d-\gamma}}{\sqrt{V_u V_t}} \left[ \frac{4\kappa e^{-0.5\kappa(t-u)}}{\sigma^2(1-e^{-\kappa(t-u)})} \right]
\]
3) Recover \( \int_u^t \sqrt{V_s} dW_s^{(1)} \) from the above equation given \( V_t, \ V_u \) and \( \int_u^t V_s ds \)

\[
\int_u^t \sqrt{V_s} dW_s^{(1)} = \left( \frac{1}{\sigma} \right) \left( V_t - V_u - \kappa \theta(t - u) + \kappa \int_u^t V_s ds \right)
\]
4) Generate a sample from the distribution of $F_t$ given

\[ \int_u^t \sqrt{V_s} dW_s^{(1)} \text{ and } \int_u^t V_s ds \]

- Conditional on $\int_u^t \sqrt{V_s} dW_s^{(1)}$ and $\int_u^t V_s ds$ $F_t$ is lognormal

with drift rate

\[ \hat{\mu} = -\frac{1}{2t} \int_u^t V_s ds + \frac{\rho}{t} \int_u^t \sqrt{V_s} dW_s^{(1)} \]

and volatility

\[ \hat{\sigma} = \sqrt{\frac{1-\rho^2}{t}} \sqrt{\int_u^t V_s ds} \]
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An *Almost* Exact Simulation Method

- The Exact Simulation Method is only appropriate for options which depend on only one future point in time because the characteristic function otherwise depends on three variables, $a$, $V_t$ and $V_u$. We cannot efficiently cache the characteristic function in three dimensions.

- If we look closely at the characteristic function we notice that it depends on $V_t$ and $V_u$ via their arithmetic and geometric mean which, at least in expectation, are approximately equal…
The characteristic function is given by

$$
\Phi(a; V_u, V_t) = \frac{\gamma(a) e^{-\frac{1}{2}(\gamma(a) - \kappa)(t-u)}(1-e^{-\kappa(t-u)})}{\kappa(1-e^{-\gamma(a)(t-u)})} \\
\times \exp \left\{ \frac{2}{\sigma^2} \left( \frac{V_u + V_t}{2} \right) \left[ \kappa(1+e^{-\kappa(t-u)}) \right] - \frac{\gamma(a)(1+e^{-\gamma(a)(t-u)})}{1-e^{-\gamma(a)(t-u)}} \right\} \\
\times I_{0.5d-1} \left[ \sqrt{\frac{V_u V_t}{\sigma^2(1-e^{-\gamma(a)(t-u)})}} \right] \\
\times I_{0.5d-1} \left[ \sqrt{\frac{4\kappa e^{-0.5\kappa(t-u)}}{\sigma^2(1-e^{-\kappa(t-u)})}} \right]
$$

Arithmetic mean

Geometric mean
An Almost Exact Simulation Method (III)

• The idea is to replace both the geometric mean and the arithmetic mean by a weighted average of the two

\[ z = \omega \frac{1}{2} (V_u + V_t) + (1 - \omega) \sqrt{V_u V_t} \]

• If the method is a good one we should find that the results are virtually independent of \( \omega \).

• In fact, observed differences for values of \( \omega \) between 0 and 1 were in the order of a fraction of a basis point for options up to 5 years maturity.
An Almost Exact Simulation Method (IV)

The characteristic function is now given by

$$\Phi(a, z) = \frac{\gamma(a) e^{-\frac{1}{2}(\gamma(a) - \kappa)(t-u)} (1 - e^{-\kappa(t-u)})}{\kappa(1 - e^{-\gamma(a)(t-u)})}$$

$$\times \exp \left\{ \frac{2}{\sigma^2} \left[ \frac{\kappa(1 + e^{-\kappa(t-u)})}{1 - e^{-\kappa(t-u)}} - \frac{\gamma(a)(1 + e^{-\gamma(a)(t-u)})}{1 - e^{-\gamma(a)(t-u)}} \right] \right\}$$

$$I_{0.5d-1} \left[ \frac{4\gamma(a) e^{-0.5\gamma(a)(t-u)}}{\sigma^2(1 - e^{-\gamma(a)(t-u)})} \right]$$

$$\times I_{0.5d-1} \left[ \frac{4\kappa e^{-0.5\kappa(t-u)}}{\sigma^2(1 - e^{-\kappa(t-u)})} \right]$$

and can be efficiently cached in two dimensions.
An Almost Exact Simulation Method (V)

• We have managed to take the heavy work out of the Monte Carlo loop. Now the overhead does not depend on the number of simulations.

• Most importantly, we can now apply the method to options which depend on several points in time (very common in Equity).

• We can improve our approximation by using knowledge of the first two moments of the true distribution. (Requires only a few evaluations of the characteristic function.)

• We use the true and approximate means and variances to shift and scale our approximate distribution. In this way we ensure to match the first two moments of the true distribution.
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Conclusions and Extensions

• We can calibrate the time dependent Heston model as efficiently as the flat Heston model.

• The Almost Exact Simulation method outperforms other published discretisation methods. It has virtually no bias. Techniques for simulating the Greeks can be used unchanged.

• It is relatively straightforward to apply the Almost Exact Simulation method to some other well known jump processes.
References

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