An “Almost Exact” Simulation Method for the Heston Model

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Madrid, October 18, 2006

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The Heston Model

$$dF_t = \sqrt{V_t}\, F_t\, dW_1$$

$$dV_t = \underbrace{\kappa_t\left(\theta_t - V_t\right)dt}_{\text{mean reverting term}} + \sigma_t \sqrt{V_t}\, dW_2$$

$$dW_1 dW_2 = \rho_t\, dt$$

- V_t: Variance (not volatility)
- σ_t: Volatility of variance
- θ_t: Long term variance
- κ_t: Mean reversion rate
- ρ_t: Correlation between changes in variance and forward

- **In Fourier Space, two dimensional Heston PDE becomes one dimensional**

=> Vanilla prices have closed form solutions

The Heston Model (II)

- Pricing involves numerically integrating analytical solutions of Riccati Equations.

- Time dependent case (with piecewise constant parameters) is no more complicated than the flat case!!!

- The Riccati Equations must be solved in sections of time in which the parameters are constant, working backwards from maturity.

- Computational time ≈ number of time sections * time to calculate flat Heston.

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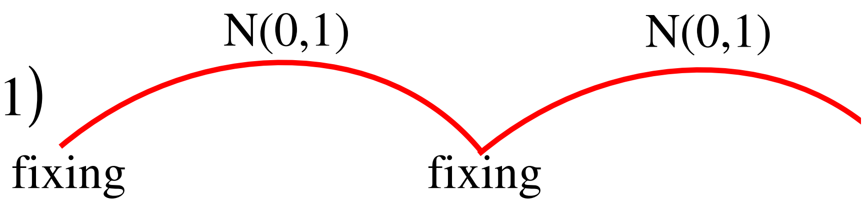
An Exact Simulation Method

- **Black Scholes has analytical solution for SDE**
  => we can "long jump" in Monte Carlo

$$S_t = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}N(0,1)}$$



- **In Stochastic Vol Models we have to do many little timesteps (SLOW!)**

- **Much research on better discretisation methods than Euler (e.g. Predictor-Corrector, Milstein…)**

- **Broadie and Kaya presented an "Exact Simulation Method" in their paper of 2003**

An Exact Simulation Method (II)

Advantages of an exact simulation method are

- **We need only simulate the dates of importance for the financial product we are modeling.**
- **The simulations have no bias in the error so we can construct valid Confidence Intervals**
- **The Order of Convergence improves from ⅓ to ½**
- **Greeks can be calculated using the same "tricks" as with Black-Scholes model because the forward price is lognormal conditional upon the path of the variance**

An Exact Simulation Method (III)

$$F_t = F_u \exp\left[-\frac{1}{2}\int_u^t V_s ds + \rho\int_u^t \sqrt{V_s}dW_s^{(1)} + \sqrt{1-\rho^2}\int_u^t \sqrt{V_s}dW_s^{(2)}\right]$$

$$V_t = V_u + \kappa\theta(t-u) - \kappa\int_u^t V_s ds + \sigma\int_u^t \sqrt{V_s}dW_s^{(1)}$$

$$dW_s^{(1)}dW_s^{(2)} = 0$$

- **Generate a sample from the distribution of V_t given V_u**
- **Generate a sample from the distribution of $\int_u^t V_s ds$ given V_t and V_u**
- **Recover $\int_u^t \sqrt{V_s}dW_s^{(1)}$ from the above equation given V_t, V_u and $\int_u^t V_s ds$**
- **Generate a sample from the distribution of F_t given $\int_u^t \sqrt{V_s}dW_s^{(1)}$ and $\int_u^t V_s ds$**

An Exact Simulation Method (IV)

1) Generate a sample from the distribution of V_t given V_u

- The distribution is known and is a non-central chi-squared distribution

- Samples can be generated using combinations of Poisson, Gamma and Normal variates

- Sankaran (1963) shows that one can transform a non-central chi squared variate into an approximately normal variate. We use this to calculate confidence intervals for V_t which we use in the next section.

An Exact Simulation Method (V)

2) Generate a sample from the distribution of $\int_u^t V_s ds$ given V_t and V_u

- Brute force method: the CDF is given by a numerical integral of the known characteristic function of the distribution. We can generate samples from uniform deviates by inverting the CDF using Newton's method.

- The characteristic function is computationally expensive as it involves modified Bessel functions of the first kind with a complex argument

- We should think about caching as much as we can!

An Exact Simulation Method (VI)

The characteristic function is given by

$$\Phi(a;V_u,V_t) = \frac{\gamma(a)e^{-\frac{1}{2}(\gamma(a)-\kappa)(t-u)}\left(1-e^{-\kappa(t-u)}\right)}{\kappa\left(1-e^{-\gamma(a)(t-u)}\right)}$$

$$\times\exp\left\{\frac{V_u+V_t}{\sigma^2}\left[\frac{\kappa\left(1+e^{-\kappa(t-u)}\right)}{1-e^{-\kappa(t-u)}}-\frac{\gamma(a)\left(1+e^{-\gamma(a)(t-u)}\right)}{1-e^{-\gamma(a)(t-u)}}\right]\right\}$$

$$\times\frac{I_{0.5d-1}\left[\sqrt{V_uV_t}\,\frac{4\gamma(a)e^{-0.5\gamma(a)(t-u)}}{\sigma^2\left(1-e^{-\gamma(a)(t-u)}\right)}\right]}{I_{0.5d-1}\left[\sqrt{V_uV_t}\,\frac{4\kappa e^{-0.5\kappa(t-u)}}{\sigma^2\left(1-e^{-\kappa(t-u)}\right)}\right]}$$

Bessel functions of the first kind with complex argument

An Exact Simulation Method (VII)

3) Recover $\int_u^t \sqrt{V_s}\,dW_s^{(1)}$ from the above equation given

V_t, V_u **and** $\int_u^t V_s\,ds$

$$\int_u^t \sqrt{V_s}\,dW_s^{(1)} = \left(\frac{1}{\sigma}\right)\left(V_t - V_u - \kappa\theta(t-u) + \kappa\int_u^t V_s\,ds\right)$$

An Exact Simulation Method (VIII)

4) Generate a sample from the distribution of F_t given

$\int_u^t \sqrt{V_s}\, dW_s^{(1)}$ and $\int_u^t V_s\, ds$

- Conditional on $\int_u^t \sqrt{V_s}\, dW_s^{(1)}$ and $\int_u^t V_s\, ds$ F_t is lognormal

  with drift rate

$$\hat{\mu} = -\frac{1}{2t}\int_u^t V_s\, ds + \frac{\rho}{t}\int_u^t \sqrt{V_s}\, dW_s^{(1)}$$

  and volatility

$$\hat{\sigma} = \sqrt{\frac{1-\rho^2}{t}}\sqrt{\int_u^t V_s\, ds}$$

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An *Almost* Exact Simulation Method

- **The Exact Simulation Method is only appropriate for options which depend on only one future point in time because the characteristic function otherwise depends on three variables, a, V_t and V_u. We cannot efficiently cache the characteristic function in three dimensions.**

- **If we look closely at the characteristic function we notice that it depends on V_t and V_u via their arithmetic and geometric mean which, at least in expectation, are approximately equal…**

An Almost Exact Simulation Method (II)

The characteristic function is given by

$$\Phi(a;V_u,V_t)=\frac{\gamma(a)e^{-\frac{1}{2}(\gamma(a)-\kappa)(t-u)}\left(1-e^{-\kappa(t-u)}\right)}{\kappa\left(1-e^{-\gamma(a)(t-u)}\right)}$$

$$\times\exp\left\{\frac{2}{\sigma^2}\frac{V_u+V_t}{2}\left[\frac{\kappa\left(1+e^{-\kappa(t-u)}\right)}{1-e^{-\kappa(t-u)}}-\frac{\gamma(a)\left(1+e^{-\gamma(a)(t-u)}\right)}{1-e^{-\gamma(a)(t-u)}}\right]\right\}$$

$$\times\frac{I_{0.5d-1}\left[\sqrt{V_uV_t}\frac{4\gamma(a)e^{-0.5\gamma(a)(t-u)}}{\sigma^2\left(1-e^{-\gamma(a)(t-u)}\right)}\right]}{I_{0.5d-1}\left[\sqrt{V_uV_t}\frac{4\kappa e^{-0.5\kappa(t-u)}}{\sigma^2\left(1-e^{-\kappa(t-u)}\right)}\right]}$$

Arithmetic mean

Geometric mean

An Almost Exact Simulation Method (III)

- The idea is to replace both the geometric mean and the arithmetic mean by a weighted average of the two

$$z = \omega \frac{1}{2}\left(V_u + V_t\right) + \left(1 - \omega\right)\sqrt{V_u V_t}$$

- If the method is a good one we should find that the results are virtually independent of ω.

- In fact, observed differences for values of ω between 0 and 1 were in the order of a fraction of a basis point for options up to 5 years maturity.

An Almost Exact Simulation Method (IV)

The characteristic function is now given by

$$\Phi(a;z) = \frac{\gamma(a) e^{-\frac{1}{2}(\gamma(a)-\kappa)(t-u)}\left(1-e^{-\kappa(t-u)}\right)}{\kappa\left(1-e^{-\gamma(a)(t-u)}\right)}$$

$$\times \exp\left\{\frac{2}{\sigma^2} z \left[\frac{\kappa\left(1+e^{-\kappa(t-u)}\right)}{1-e^{-\kappa(t-u)}} - \frac{\gamma(a)\left(1+e^{-\gamma(a)(t-u)}\right)}{1-e^{-\gamma(a)(t-u)}}\right]\right\}$$

$$\times \frac{I_{0.5d-1}\left[z \dfrac{4\gamma(a) e^{-0.5\gamma(a)(t-u)}}{\sigma^2\left(1-e^{-\gamma(a)(t-u)}\right)}\right]}{I_{0.5d-1}\left[z \dfrac{4\kappa e^{-0.5\kappa(t-u)}}{\sigma^2\left(1-e^{-\kappa(t-u)}\right)}\right]}$$

and can be efficiently cached in two dimensions.

An Almost Exact Simulation Method (V)

- We have managed to take the heavy work out of the Monte Carlo loop. Now the overhead does not depend on the number of simulations.
- Most importantly, we can now apply the method to options which depend on several points in time (very common in Equity).
- We can improve our approximation by using knowledge of the first two moments of the true distribution. (Requires only a few evaluations of the characteristic function.)
- We use the true and approximate means and variances to shift and scale our approximate distribution. In this way we ensure to match the first two moments of the true distribution.

Contents

Conclusions and Extensions

- We can calibrate the time dependent Heston model as efficiently as the flat Heston model.
- The Almost Exact Simulation method outperforms other published discretisation methods. It has virtually no bias. Techniques for simulating the Greeks can be used unchanged.
- It is relatively straightforward to apply the Almost Exact Simulation method to some other well known jump processes.

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