A hybrid optimization strategy for portfolio selection

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Harry M. Markowitz (Nobel Prize, 1990)

" An investor who knew future returns with certainty would invest in only one security, namely the one with the highest future return. If several securities had the same, highest, future return then the investor would be indifferent between any of these, or any combination of these. In no case would the investor actually prefer a diversified portfolio. But diversification is a common and reasonable investment practice. Why? To reduce uncertainty! Clearly, the existence of **uncertainty is essential to the analysis of rational investment behavior**[…]

It seemed obvious that investors are concerned with **risk** and **return**, and that these should be measured for the portfolio as a whole. **Variance** (or, equivalently, standard deviation), came to mind **as a measure of risk** of the portfolio. The fact that the variance of the portfolio, that is the variance of a weighted sum, involved all covariance terms added to the plausibility of the approach. Since there were two criteria - expected return and risk – the natural approach for an economics student was to imagine the investor selecting a point from the set of **Pareto optimal expected return, variance of return combinations**, now known as the **efficient frontier**. […] "

Excerpt from FOUNDATIONS OF PORTFOLIO THEORY, Nobel Lecture, December 7, 1990

Portfolio optimization

Maximize returns / minimize risks

Multiobjective optimization problem

- For a given expected return find the investment that minimizes the risk of the portfolio

⇕ Dual optimization problems

- For a given risk find the investment that maximizes the expected return of the portfolio

Definitions

- Values / returns of the products at time *t:*

$$S_i(t); \qquad r_i(t) = \frac{S_i(t+\Delta t) - S_i(t)}{S_i(t)}; \qquad i = 1,\dots,n$$

- Assuming the investments $\{c_i(t)\}_{i=1}^n$ that define the portfolio do not change in the interval $[t, t+\Delta t]$

$$P(t) = \sum_{i=1}^{n} c_i(t) S_i(t);$$

$$r_P(t) = \frac{P(t+\Delta t) - P(t)}{P(t)} = \sum_{i=1}^{n} w_i(t) r_i(t); \qquad w_i(t) = \frac{c_i(t) S_i(t)}{\sum_{j=1}^{n} c_j(t) S_j(t)}$$

Assumptions

- Returns $\{r_i(t)\}_{i=1}^{n}$ are **random variables**.
- Estimates of the pdf of these random variables are available (e.g. estimated from values in recent history).

$$< \mathbf{r} > = mean\left\{ \mathbf{r}(t) \right\} \qquad \mathbf{\Sigma} = \mathrm{var}\left\{ \mathbf{r}(t)\, \mathbf{r}(t)^{+} \right\}$$

- Expected return of the portfolio in the interval $[t, t+\Delta t]$

$$< r_P(t) > = \sum_{i=1}^{n} w_i < r_i(t) > = \mathbf{w}^{+} \cdot < \mathbf{r} >;$$

$$w_i = \frac{c_i(t) S_i(t)}{\sum_{j=1}^{n} c_j(t) S_j(t)}; \qquad 0 \le w_i(t) \le 1; \qquad \sum_{i=1}^{n} w_i(t) = 1$$

Markowitz model

Variance of the portfolio as a measure of **risk**

Min $\quad \mathrm{var}(r_P) = \mathbf{w}^+ \cdot \mathbf{\Sigma} \cdot \mathbf{w}$

s.t. $\quad \mathbf{w}^+ \cdot \langle \mathbf{r} \rangle = R^*$

$$\sum_{i=1}^{n} w_i = 1$$

$$w_i \geq 0 \,, \quad i = 1, \ldots, n$$

Max $\quad \mathbf{w}^+ \cdot \langle \mathbf{r} \rangle$

s.t. $\quad \mathbf{w}^+ \cdot \mathbf{\Sigma} \cdot \mathbf{w} = \sigma^*$

$$\sum_{i=1}^{n} w_i = 1$$

$$w_i \geq 0 \,, \quad i = 1, \ldots, n$$

Dual optimization problems

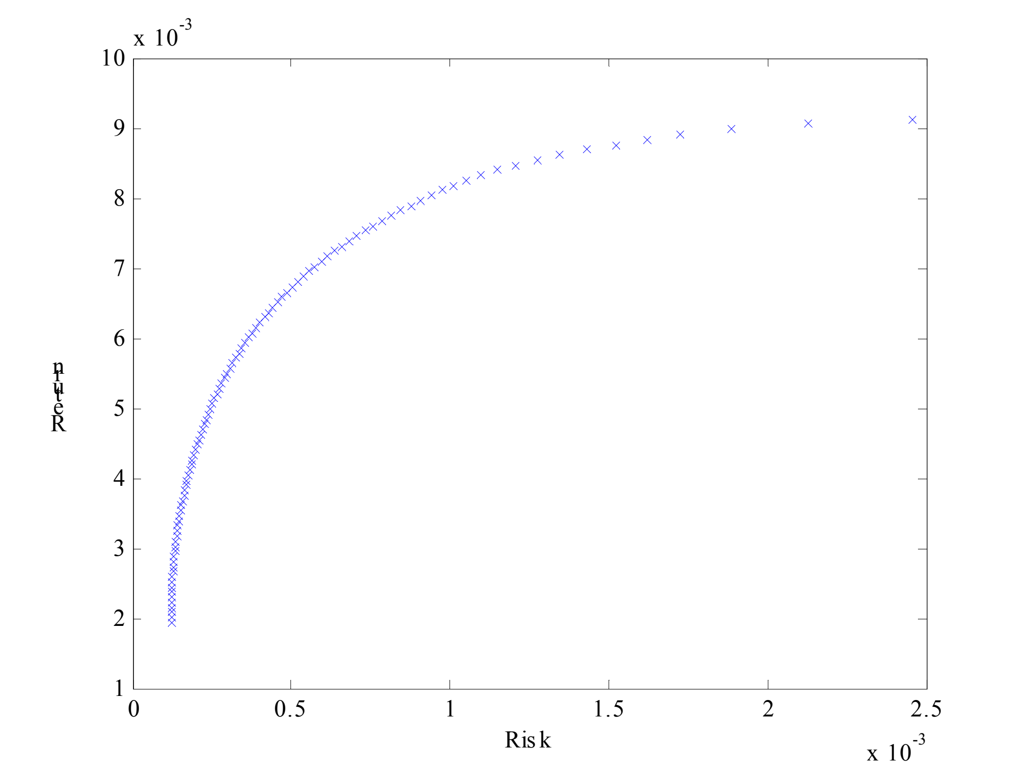
Minimize risk for a given expected return R*

Maximize the expected return for a given value of the risk measure, σ*

Quadratic objective function + Linear restrictions➔ **Quadratic programming**

Efficient frontier

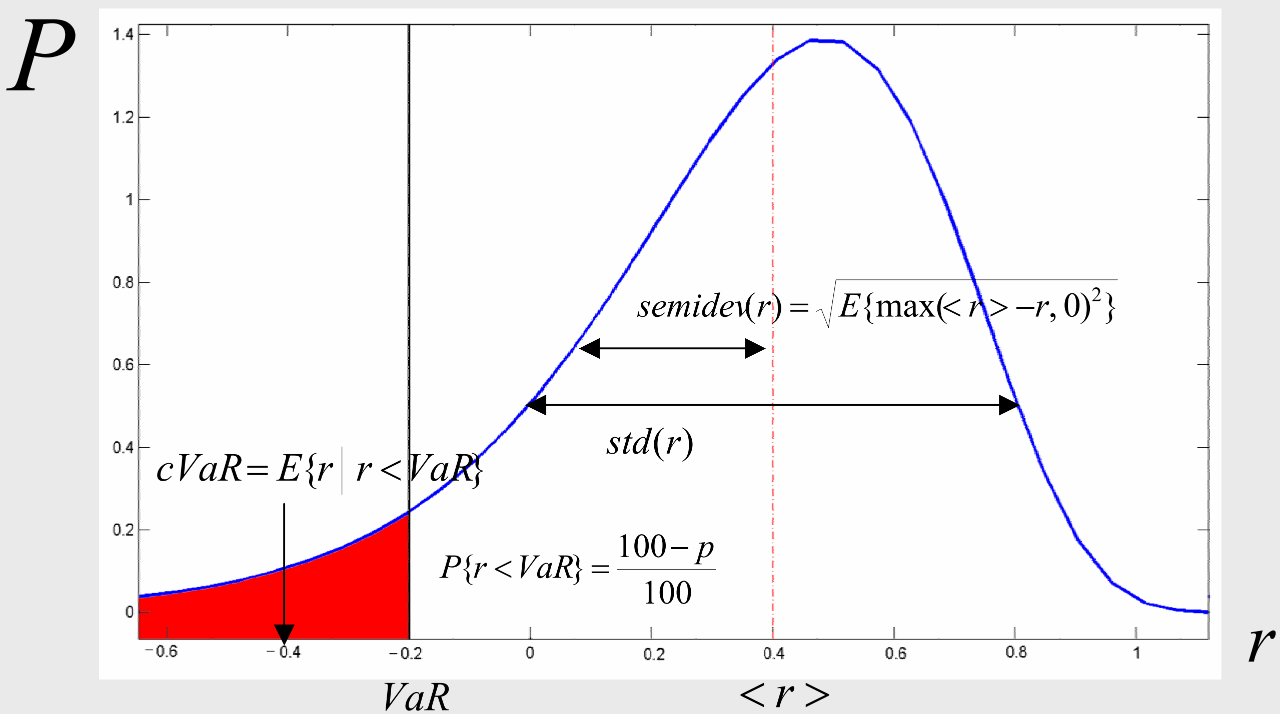
Optimal portfolios constructed by investment in 98 assets included in the idex S&P 100



Extensions of the Markowitz model (I)

- Other measures of risk:

Semideviation, VaR, ShortFall (conditional VaR), etc.



Extensions of the Markowitz model (II)

- Realistic **constraints**
  1. Concentration of capital constraints [linear]

     $$\mathbf{l} \leq \mathbf{A} \cdot \mathbf{w} \leq \mathbf{u} \qquad \text{[Example: } 0.1 \leq w_1 + w_3 \leq 0.3 \text{]}$$

  2. Minimum / maximum investment on a given asset [linear]

     $$a_i \leq w_i \leq b_i \ , \quad a_i, b_i \geq 0 \quad i = 1, \ldots, n$$

  3. **Cardinality constraints** [non-linear]

     $$\left| \left\{ i \in \{1, \ldots, n\} : w_i \neq 0 \right\} \right| \leq c$$

- Combination (2)+(3)

$$\sum_{i=1}^{n} z_i \leq c \quad , \quad z_i \in \{0, 1\}, \quad a_i z_i \leq w_i \leq b_i z_i \ , \quad a_i, b_i \geq 0 \quad i = 1, \ldots, n$$

Markowitz model + restrictions

Min $\mathbf{w}^{+} \cdot \mathbf{\Sigma} \cdot \mathbf{w}$

s.t. $\mathbf{w}^{+} \cdot <\mathbf{r}> = R^{*}$

$$\sum_{i=1}^{n} w_i = 1$$

$$w_i \geq 0 \,, \quad i = 1,\ldots,n$$

$$\mathbf{l} \leq \mathbf{A} \cdot \mathbf{w} \leq \mathbf{u}$$

$$\sum_{i=1}^{n} z_i \leq c \quad , \quad z_i \in \{0,1\}$$

$$a_i z_i \leq w_i \leq b_i z_i \,, \quad a_i, b_i \geq 0 \quad i = 1,\ldots,n$$

Mixed combinatorial-quadratic problem

1. Quadratic optimization problem:

   Given the values $\{z_i\}_{i=1}^{n}$, the problem involves optimizing a quadratic function with linear restrictions.

   Quadratic programming

2. Combinatorial optimization problem

   Find the values $\{z_i\}_{i=1}^{n}$ for which the solution of the quadratic optimization problem defined in (1) is optimal.

   Simulated annealing / genetic algorithm

Combinatorial optimization problem

- **Exhaustive search**

  Example: n = 100, m = 10 $\binom{m}{n} = \binom{100}{10} = 1.73 \cdot 10^{13}$

- Branch-and-bound
- **Genetic algorithms**
- **Simulated annealing**
- Tabu search
- Ant-colony optimization
- …

Optimization in the natural sciences

- **Simulated annealing** (Phyics, metallurgy)

  "Molten substances, when cooled, form crystalline structures that correspond to minima of the internal energy of the system".

- **Genetic algorithms and evolution strategies** (biology).

  "Evolution by the combination of random changes and selection leads to the appearance of structures that maximize fitness".

- **Ant-colony optimization** (etiology, complex systems)

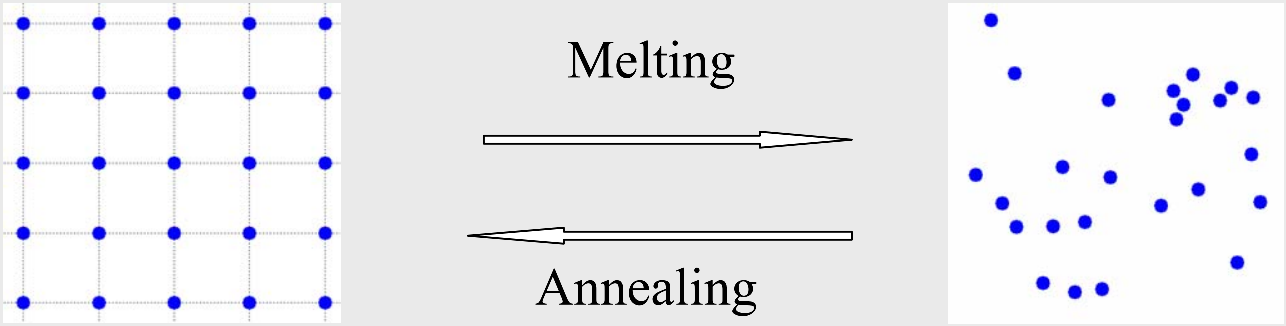
  "Emergent behavior in systems with agents that interact locally may be optimal in a global manner".

Simulated annealing (physics)

- The state of a substance is determined by minimizing its Gibbs free energy

$$G = E - T\,S$$

  - **Internal energy** term: Dominant at low temperature.
  - **Entropic term** : Dominant at high temperature.

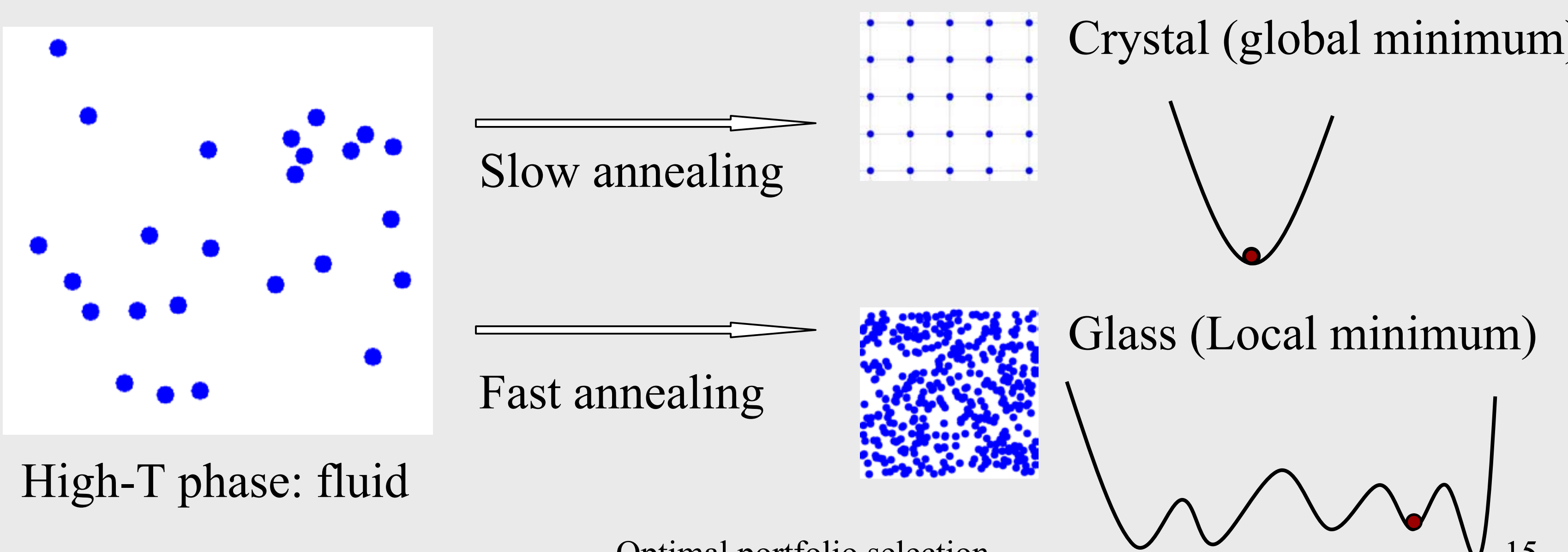


Low T- solution (E min)
(solid state)

High T- solution (G min)
(fluid state)

Annealing

1. Begin with a melted solid.
2. Lower the temperature gradually.
3. At T = 0 we reach a state corresponding to a minimum of E (if annealing suffuciently slow, the global minimum).



Annealing for optimization

1. Start with an initial configuration **$\{c_i, E_i\}$** $:= \{c_0, E_0\}$, at temperature $T := T_0$
2. Repeat

   2.1 Repeat [annealing epoch at temperature T]

   - Generate a configuration $\{c_j, E_j\}$ from the **neighborhood** of c_i
   - Accept/reject new configuration

     $\{c_i, E_i\} := \{c_j, E_j\}$ w. prob. $\min[1, \exp\{-(E_j - E_i)/T\}]$

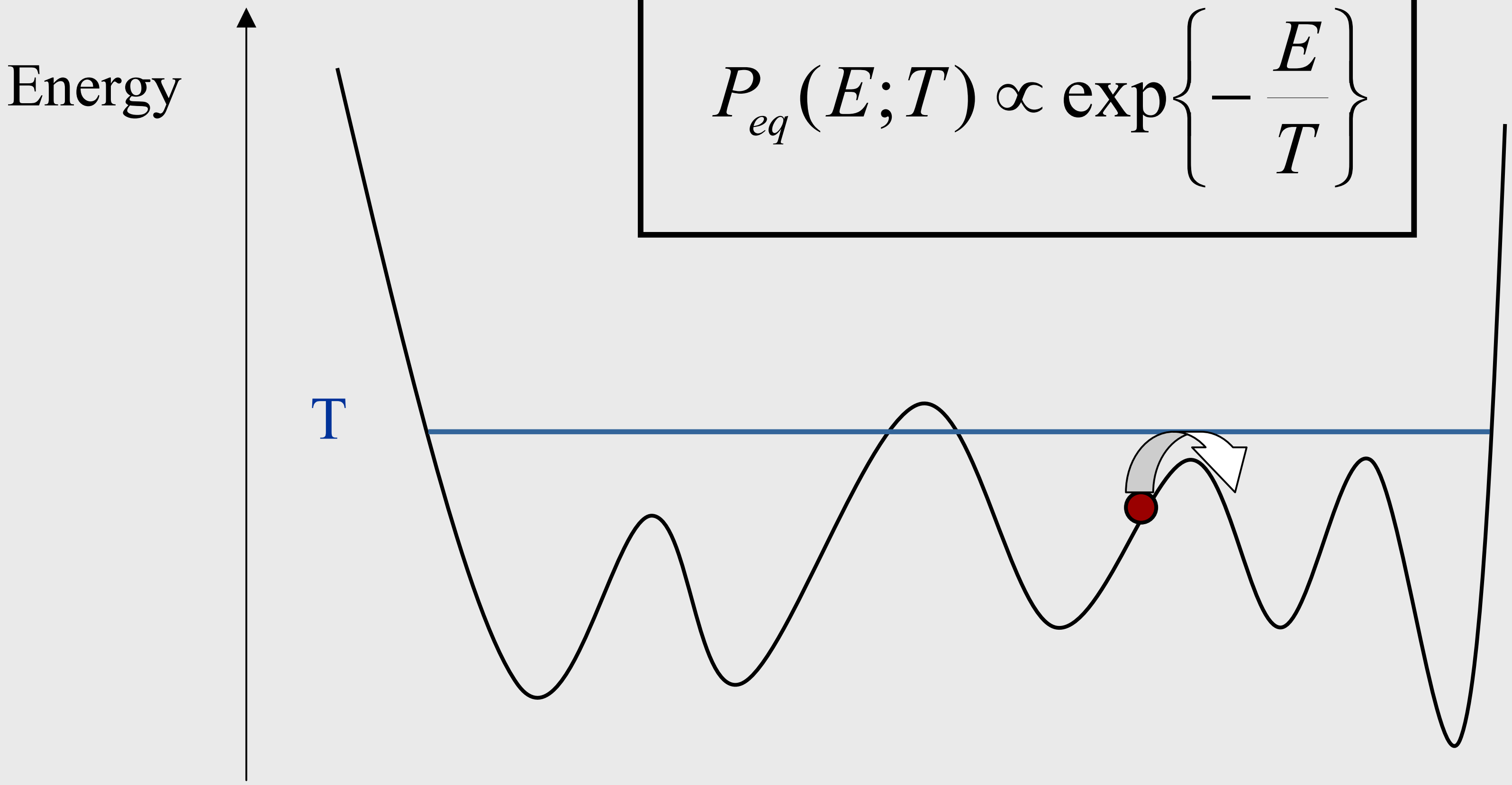
   Until local equilibrium at temperature T is reached

   2.2 Lower T [**annealing schedule**]

   Until convergence

Local equilibrium

$$P_{eq}(E;T) \propto \exp\left\{-\frac{E}{T}\right\}$$



Implementation of Simulated Annealing

- Configuration (c): Select of **m** from the **n** products.

  Example: n = 5, m = 3

  Portfolio formed by investment in assets labeled {2,3,5}

IN			OUT	
2	3	5	1	4

- Cost function (E): Risk associated with the optimal portfolio that is a solution of a quadratic optimization problem involving only the m products selected.
- Annealing schedule: Geometric annealing

$$T := \alpha T, \qquad \alpha \in [0.8, 0.99]$$

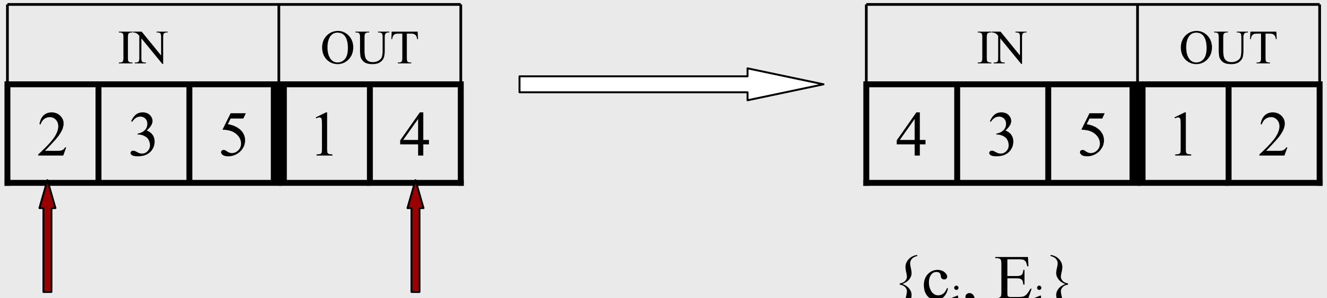
Neighborhood

- Initial configuration: Random permutation vector

$\{c_i, E_i\} := \{c_0, E_0\}$

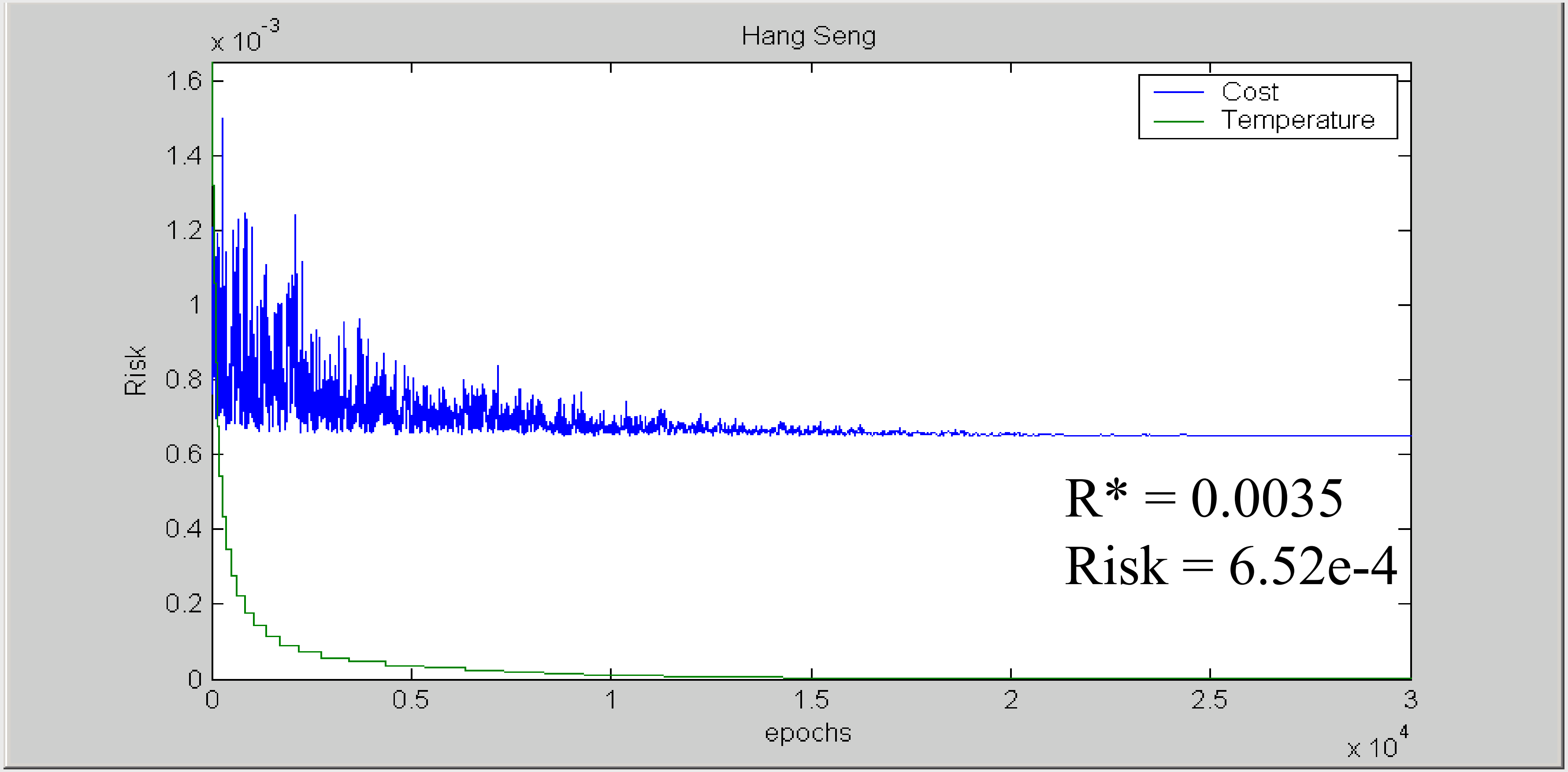
IN			OUT	
2	3	5	1	4

- Generation of candidate solution: Exchange two products between the IN and OUT parts of the permutation vector.



$\{c_j, E_j\}$

Hang Seng portfolio (10 of 31 assets)



Genetic algorithm (biology)

- Evolution: Random innovations + selection
  1. Generate a **population** of P individuals (chromosomes)
  2. Repeat

     2.1 Create parent set by random selection of n_P individuals with replacement from the population.

     2.2 Repeat [INNOVATION]
     - Select two individuals at random.
     - Apply **crossover** operator to generate n_C children
     - Apply **mutation** operator to each of the n_C children

     Until parent set is empty

     2.3 Extend population with $n_C n_P/2$ children generated.

     2.4 **Select** P individuals from the population. [SELECTION]

     Until convergence

Bitstring representation

- Chromosome: Select of **m** from the **n** products.

  Example: n = 5, m = 3

  Portfolio formed by investment in assets labeled {2,3,5}

1	2	3	4	5
0	1	1	0	1

- Mutation operation: Permutation of two bits

1	2	3	4	5
0	1	1	0	1

{2,3,5}

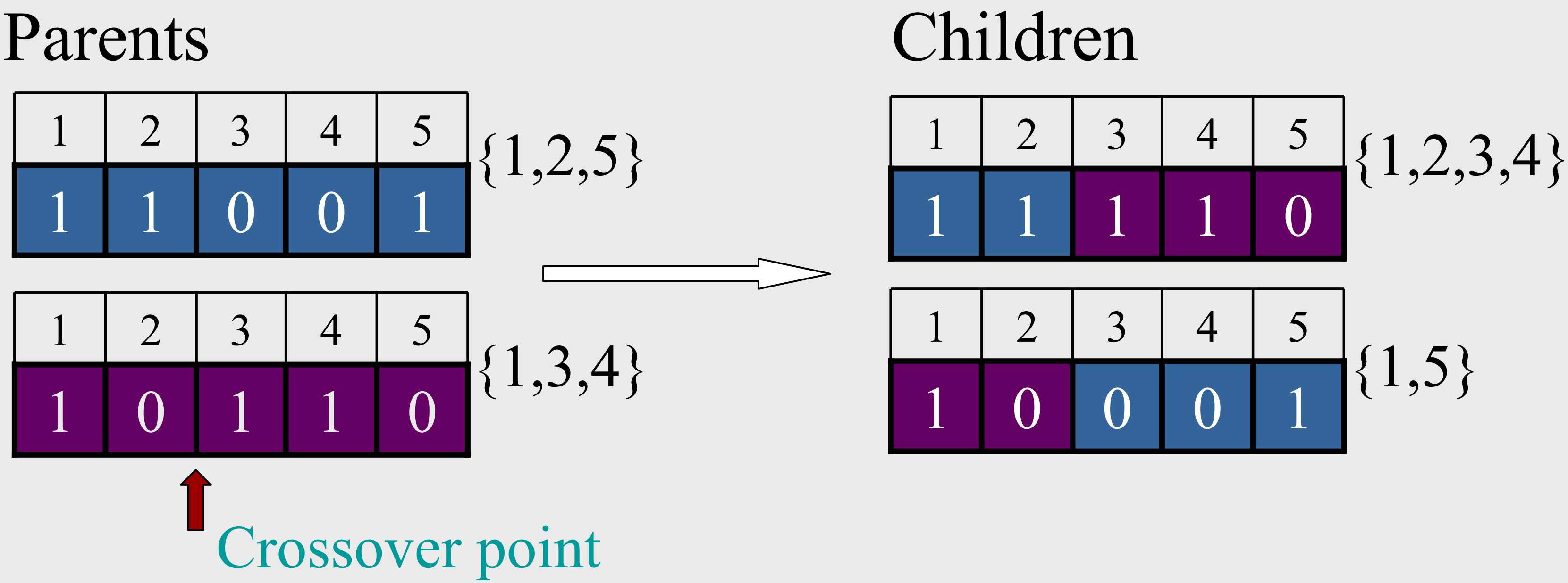
Mutation

1	2	3	4	5
1	1	0	0	1

{1,2,5}

Crossover

- One-point crossover:



- The usual crossover operators (n-point, uniform, etc.) do not preserve cardinality.

Penalties / reparation of chromosomes

- Penalties: Increase the value of the cost function for individuals that violate the cardinality constraint.
  - Death penalty: Assign individuals with the incorrect cardinality the worst possible value for the cost function.
  - Penalty functions: The amount of penalization depends on the amount by which the constraint is violated.
- Chromosome repair: Modify the value of the genes until constraint is satisfied.
  - Random repair: Genes to modify are selected at random.
  - Heuristic repair: Optimize without the constraint on cardinality and set to zero the bit-value of the genes that correspond to assets with the lowest weights in the unconstrained optimal portfolio.

Set representation: R3 crossover

A child chromosome is composed by

- Assets present in both parents.
- A random selection of the assets present in only one of the parents.

- Example ($n = 5$, $m = 3$):

Parents: **P1** $\equiv \{1,2,5\}$, **P2** $\equiv \{1,3,4\}$

Sets: **A** $\equiv P1 \cap P2 = \{1\}$, **D** $\equiv P1 \Delta P2 = \{2,3,4,5\}$

Children: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{1,4,5\}$

Problem: Overexploitation of information contained in the parents.

Set representation: RAR crossover (I)

1. Create auxiliary sets (A,B,C,D,E) whose initial composition is
   - A : Assets present in both parents.
   - B : Assets not present in any of the parents.
   - $C \equiv D$: Assets present in only one parent.
   - E : Empty set.
2. Build G:
   - w copies of elements from A and B.
   - 1 copy of elements from C and D.

Set representation: RAR crossover (II)

3. Build chromosome

 3.1 Repeat

 Extract an element from G (without replacement)

 - If the element comes from A or C, and is not an element of E, include it in the child chromosome.
 - If the element comes from B or D, include it in set E.

 Until chromosome is complete or bag G is empty

 3.2 If chromosome is not complete include assets not yet included selected at random.

RAR crossover

- Example (n = 5, m = 3, w = 2):

Parents: **P1** ≡ {1,4,5}, **P2** ≡ {1,3,4}

Sets: **A** ≡ **P1** ∩ **P2** = {1,4}, **B** ≡ {2}

C ≡ **D** ≡ **P1** Δ **P2** = {3,5}

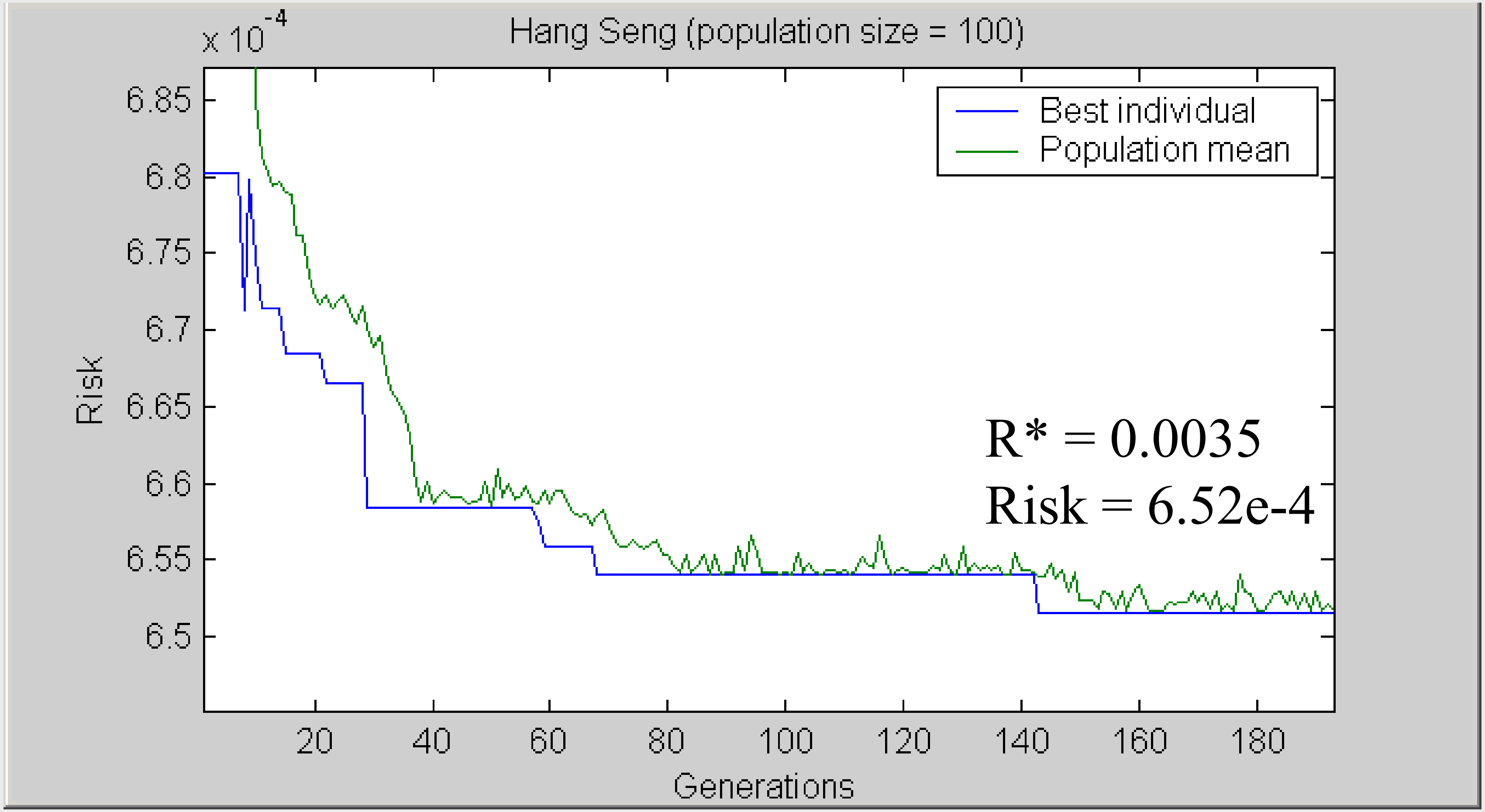
Element	**G**	**E**	**Child**
-	{1a, 1a, 4a, 4a,2,b, 2b, 3c, 5c, 3d, 5d}	{}	{}
2b	{1a, 1a, 4a, 4a, 2b, 3c, 5c, 3d, 5d}	{2}	{}
1a	{1a, 4a, 4a, 2b, 3c, 5c, 3d, 5d}	{2}	{1}
3c	{1a, 4a, 4a, 2b, 5c, 3d, 5d}	{2}	{1,3}
5c	{1a, 4a, 4a, 2b, 3d, 5d}	{2}	{1,3,5}

Implicit mutation in RAR crossover

Parents: **P1** ≡ {1,4,5}, **P2** ≡ {1,3,4}

Element	**G**	**E**	**Child**
-	{1a, 1a, 4a, 4a,2,b, 2b, 3c, 5c, 3d, 5d}	{}	{}
3d	{1a,1a,4a,4a,2b,2b,3c,5c,5d}	{3}	{}
5d	{1a,1a,4a,4a,2b,2b,3c,5c}	{3,5}	{}
2b	{1a,1a,4a,4a,2b,3c,5c}	{3,5,2}	{}
1a	{1a,4a,4a,2b,3c,5c}	{3,5,2}	{1}
4a	{1a,4a,2b,3c,5c}	{3,5,2}	{1,4}
	…		
-	{}	{3,5,2}	{1,4}
2	Select from {2,3,5}	{3,5,2}	{1,**2**,4}

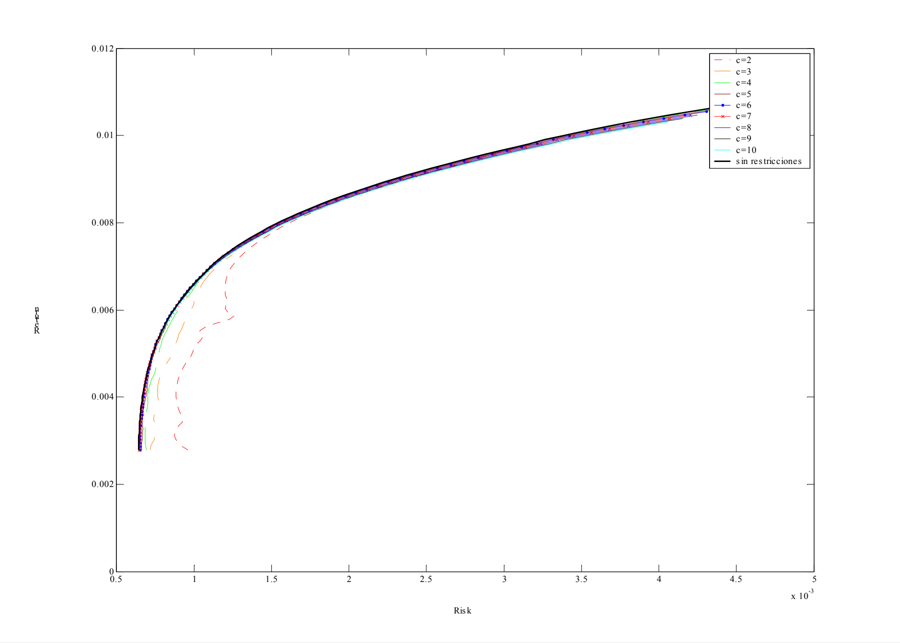
Hang Seng portfolio (10 of 31 assets)

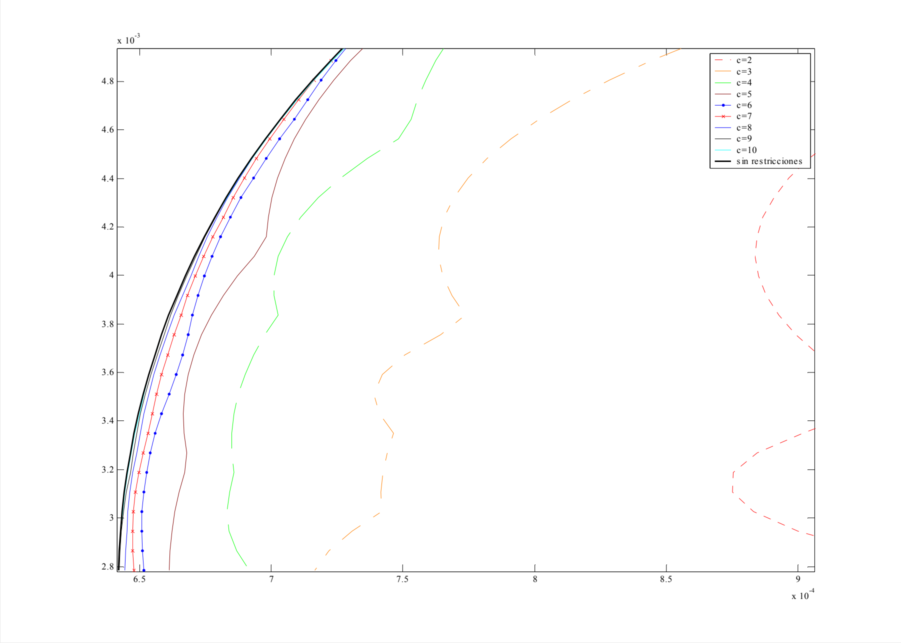


Benchmarks

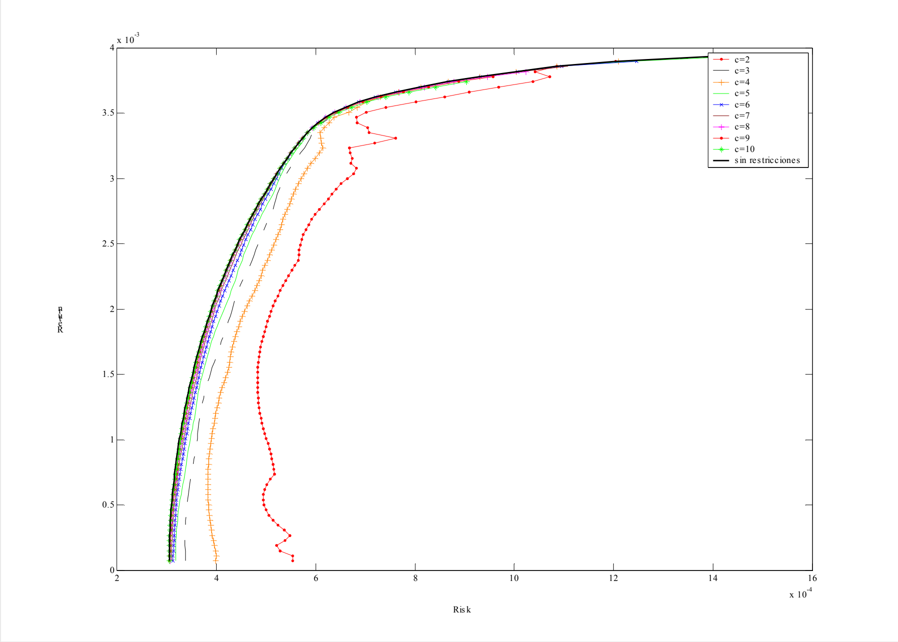
- OR-library: Historical estimates using weekly samples of expected returns and covariance matrices
  - Hang Seng (HK)
  - DAX (DE)
  - FTSE (GB)
  - SP & P (100)
  - Nikkei (JP)
- Experiments:
  - 100 points on the efficient frontier.
  - 30 executions per point.

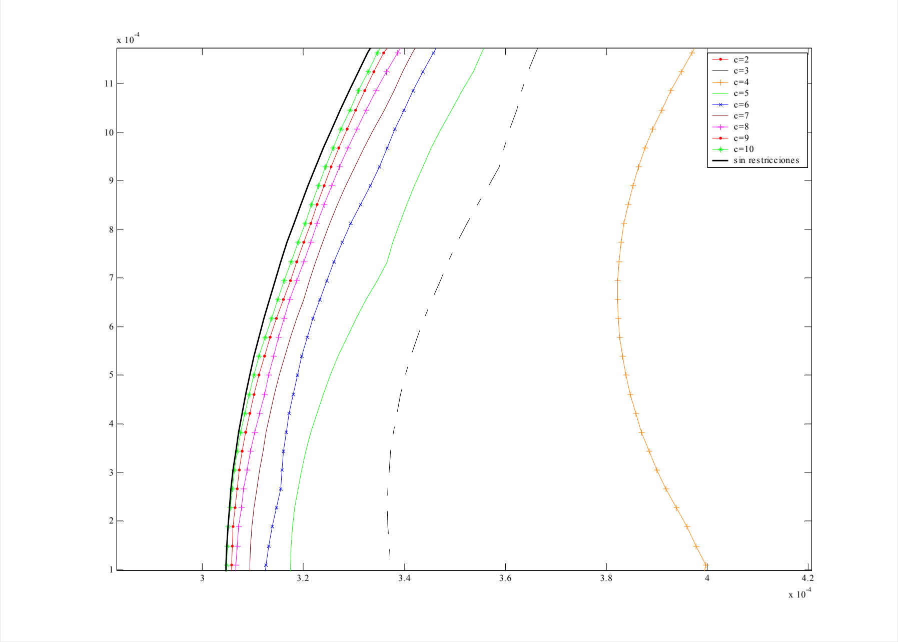
Hang Seng efficient frontier (n = 31)





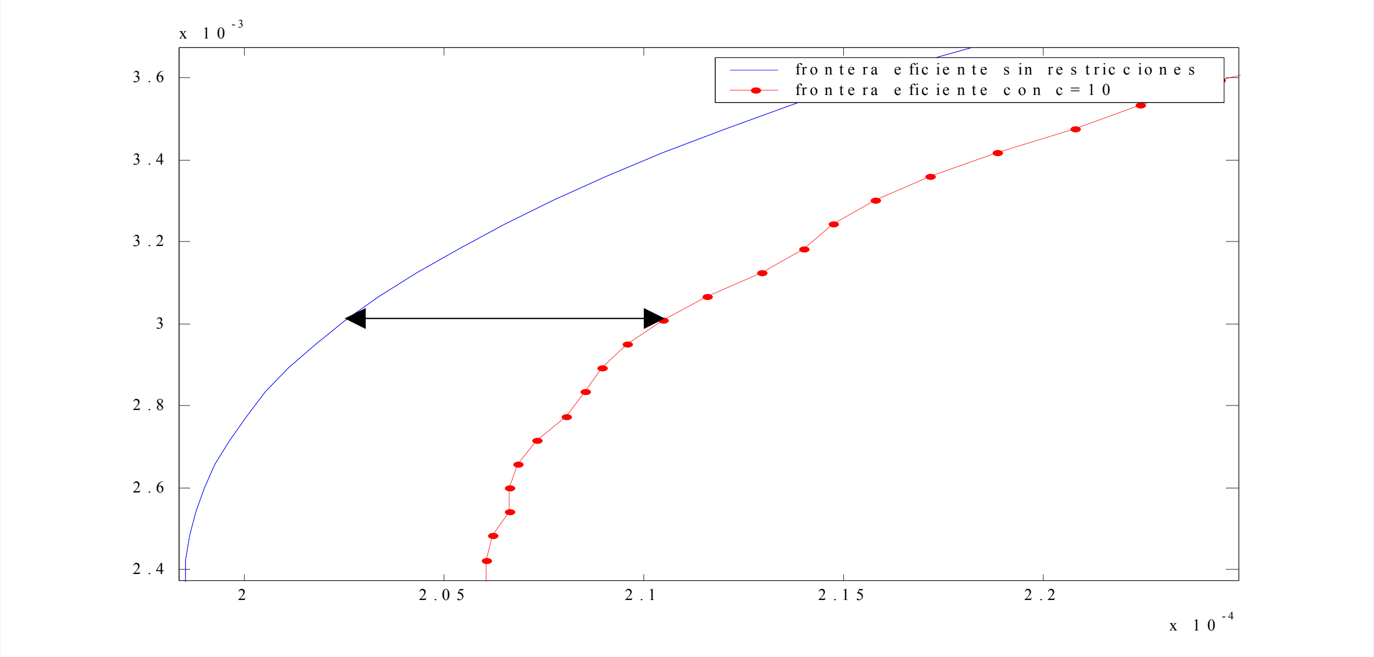
Nikkei efficient frontier (n = 225)





Error measure

Average horizontal relative distance between a point on the unconstrained efficient frontier and the corresponding point on the constrained frontier.



Error measure

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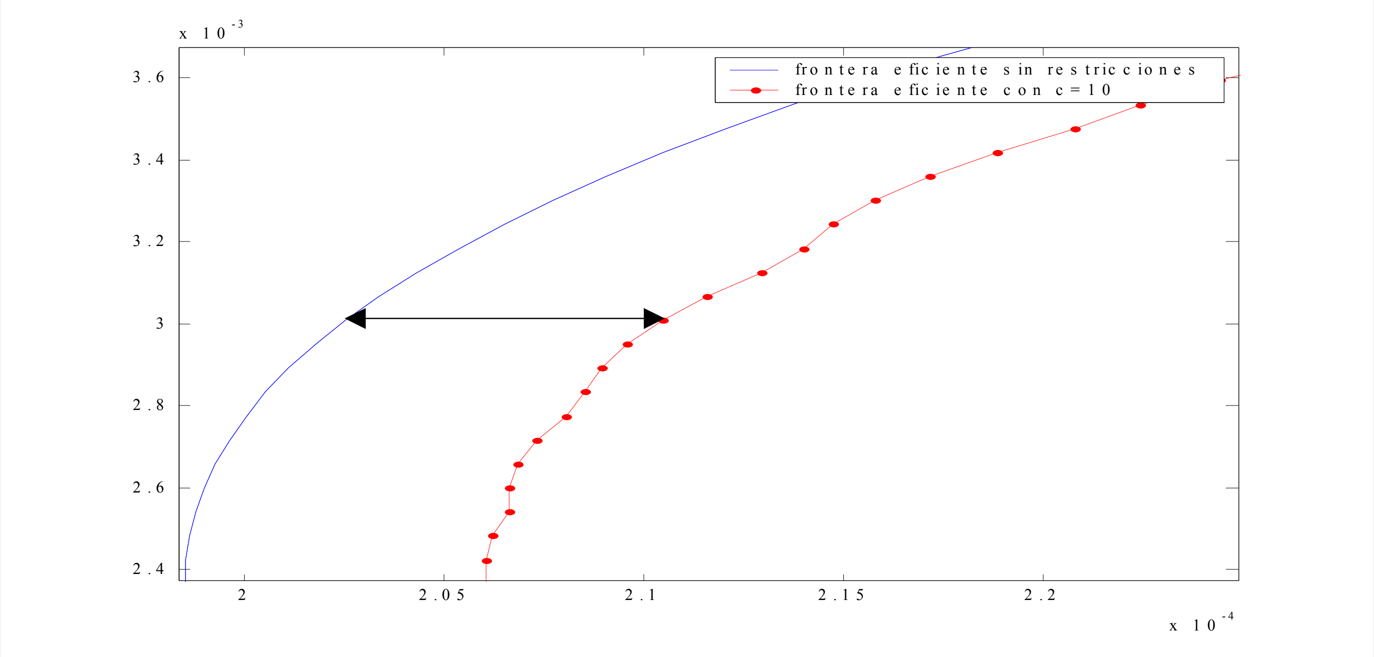

GA results (bitstring representation)

STRATEGY	INDICES	# Assets	Average (%)	Stdev	frequency of best	Time (s)
Death Penalty	Hang Seng	31	0.00324393	0.0E+00	0.98	390.2
	DAX	85	2.70352541	2.6E-03	0.86	456.0
	FTSE	89	1.94071104	4.0E-04	0.93	953.0
	S&P	98	4.76045939	1.6E-03	0.91	932.6
	Nikkei	225	0.27886285	1.7E-03	0.82	1009.5
Linear Penalty Function	Hang Seng	31	**0.00321150**	0.0E+00	1.00	353.9
	DAX	85	2.62436368	2.4E-03	0.86	420.1
	FTSE	89	1.92150019	2.4E-04	0.93	886.3
	S&P	98	4.72453770	9.3E-04	0.93	862.2
	Nikkei	225	0.26253413	2.3E-03	0.84	927.6
Quadratic Penalty Function	Hang Seng	31	0.00852899	1.1E-04	0.98	295.8
	DAX	85	2.62819331	2.8E-03	0.87	509.3
	FTSE	89	1.93368663	3.5E-04	0.93	853.0
	S&P	98	4.73613891	1.2E-03	0.92	874.5
	Nikkei	225	0.26531752	1.6E-03	0.83	943.3
Random repair	Hang Seng	31	**0.00321150**	0.0E+00	1.00	417.3
	DAX	85	2.59586957	1.6E-03	0.89	610.2
	FTSE	89	1.92600177	1.2E-04	0.97	1084.3
	S&P	98	4.70413615	3.4E-04	0.96	1091.2
	Nikkei	225	0.25047296	1.3E-03	0.86	1149.2
Heuristic repair	Hang Seng	31	**0.00321150**	0.0E+00	1.00	415.5
	DAX	85	**2.55216320**	6.8E-04	0.95	560.3
	FTSE	89	**1.92522299**	9.2E-05	0.96	1216.9
	S&P	98	**4.69979026**	1.9E-04	0.97	1205.4
	Nikkei	225	**0.23133926**	8.9E-04	0.91	1436.7

GA results (set representation)

STRATEGY	INDICES	# Assets	Average (%)	Stdev	Frequency of best	Time (s)
R3 Crossover	Hang Seng	31	0.02019022	7.1E-04	0.95	371.6
	DAX	85	4.33637863	3.4E-02	0.34	337.7
	FTSE	89	2.37012069	5.7E-03	0.40	633.0
	S&P	98	5.75089932	1.2E-02	0.36	608.2
	Nikkei	225	2.01302119	2.2E-02	0.12	638.5
RAR Crossover w=1	Hang Seng	31	**0.00321150**	0.0E+00	1.00	535.3
	DAX	85	2.53620543	2.5E-04	0.99	598.9
	FTSE	89	**1.92991152**	1.0E-04	0.94	1257.9
	S&P	98	**4.69506892**	5.2E-05	0.99	1163.7
	Nikkei	225	**0.20197748**	0.0E+00	1.00	1601.2
RAR Crossover w=2	Hang Seng	31	**0.00321150**	0.0E+00	1.00	453.8
	DAX	85	2.53187099	1.0E-05	0.99	510.9
	FTSE	89	1.93036965	1.0E-04	0.93	1076.9
	S&P	98	4.69702029	1.5E-04	0.99	1035.6
	Nikkei	225	**0.20197748**	0.0E+00	1.00	1471.8
RAR Crossover w=3	Hang Seng	31	**0.00321150**	0.0E+00	1.00	508.8
	DAX	85	**2.53180074**	4.4E-06	0.99	552.7
	FTSE	89	1.93255857	1.3E-04	0.92	1079.7
	S&P	98	4.70596057	2.9E-04	0.98	1083.2
	Nikkei	225	**0.20197748**	0.0E+00	1.00	1465.8

Results (best of 30 executions)

INDICES	Number of Assets	GA (%)	TS(*) (%)
Hang Seng	31	**0.00321150**	0.00344745
DAX	85	**2.53162860**	2.53845000
FTSE	89	**1.92150019**	1.92711000
S&P	98	**4.69373181**	4.69426000
Nikkei	225	**0.20197748**	0.20478600

(*) Schaerf, A. “Local search techniques for constrained portfolio selection problems”. Computational Economics, vol. 20 177-190 (2002)

Conclusions

- **Cardinality constraints** are difficult to handle.
- A hybrid strategy for portfolio selection:
  - Use a heuristic algorithm for the selection of the subset of assets for investment [combinatorial optimization]
  - Use the solution the optimization problem in the restricted universe to guide the search of the heuristic algorithm. [quadratic optimization]

- Heuristic methods for combinatorial optimization:
  - Simulated annealing
  - Genetic algorithm
  - Ant-colony optimization (to be done …)

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