A PDE approach to the pricing of credit derivatives.

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AGENDA

1. Define CDO, n^{th} to default and m^{th} worst performance Swaps.

2. Structural framework.

3. Probabilities needed to compute these derivatives.

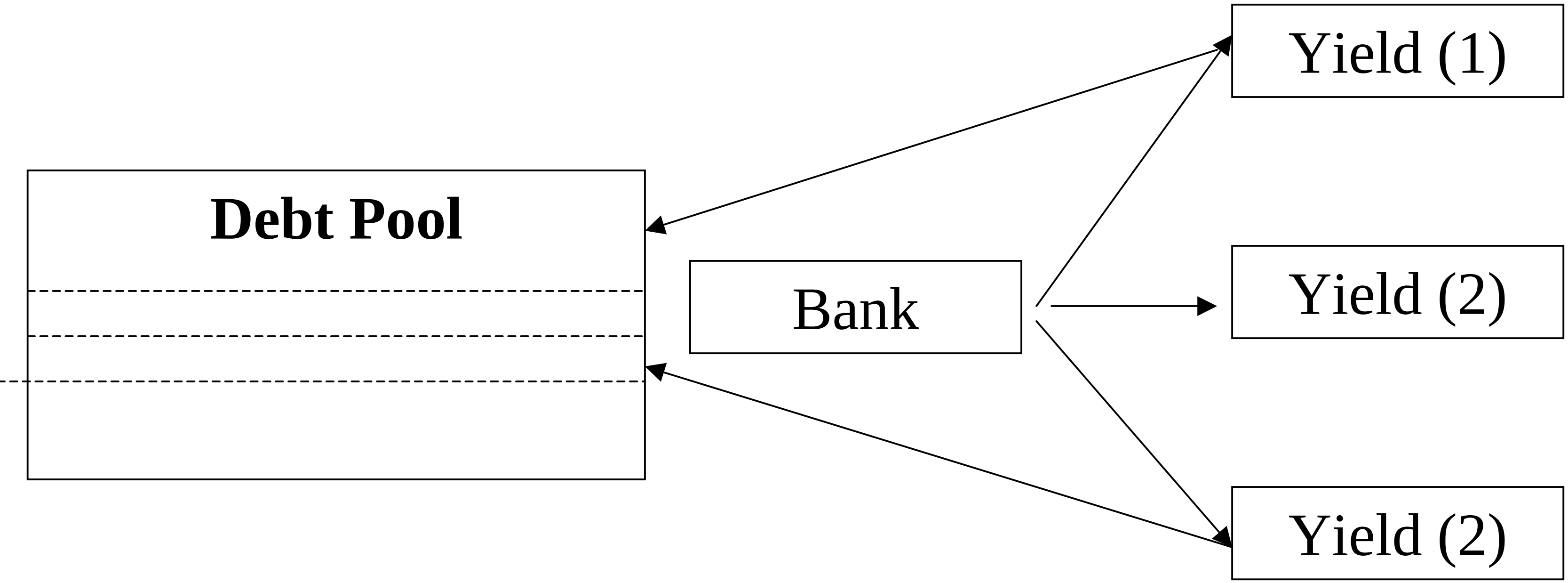
4. 2-dimensional solution and extensions.

5. Actual pricing of these derivatives.

n^{th} to default Swap

The buyer of protection pays a specified rate on a specified notional principal until the n^{th} default occurs among a specified set of N reference entities or until the end of the contract's life. The payments are usually made quarterly (we assume only an initial payment). If the n^{th} default occurs before the contract maturity, the buyer of protection can present bonds issued by the defaulting entity to the seller of protection in exchange for the face value of the bonds.

Collateralized Debt Obligation



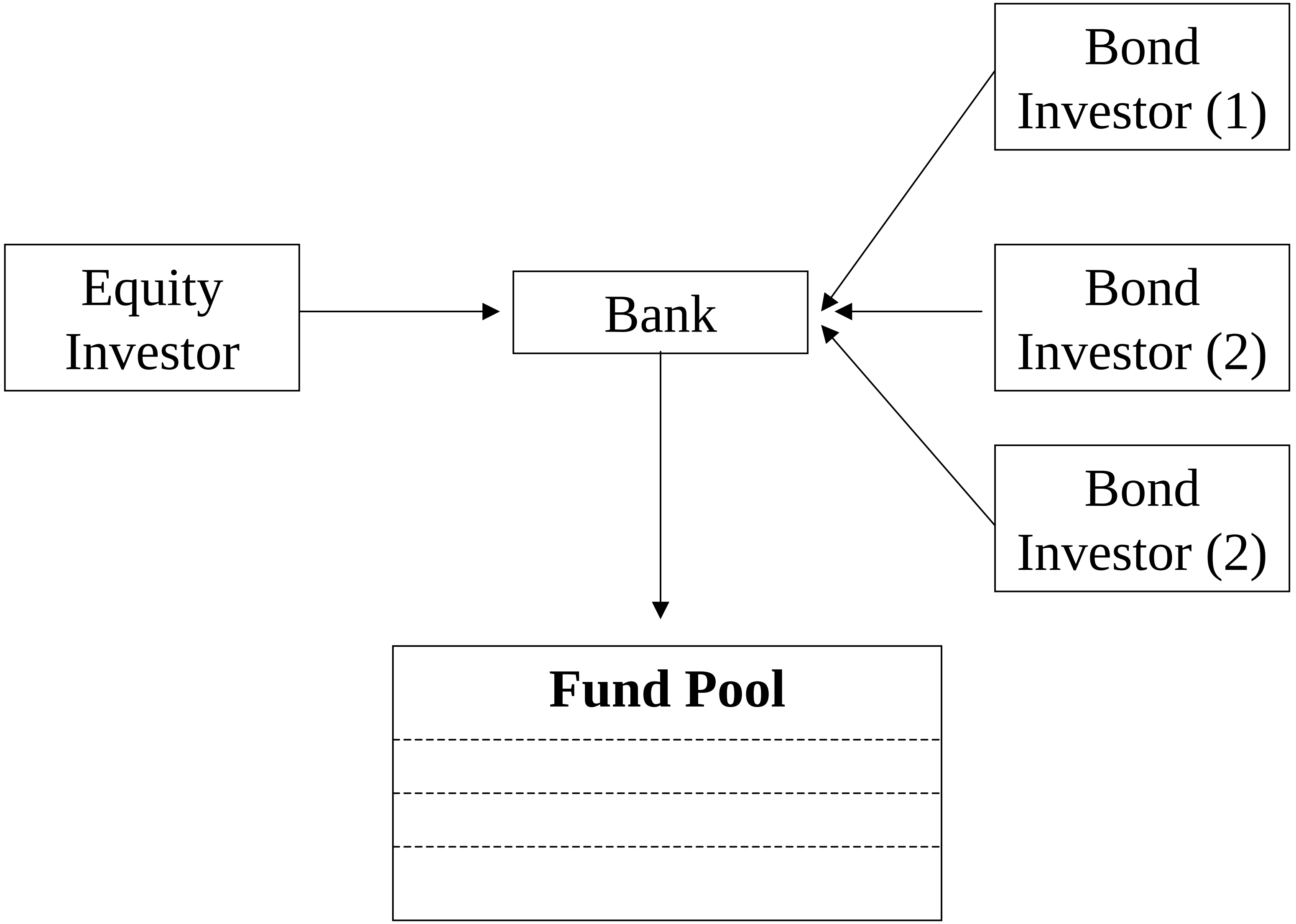
CDO's

They are a way of creating M securities from a portfolio of N debt instruments (i.e. defaultable bonds).
Security 1 (tranche): Absorbs all credit losses from the portfolio during the life of the CDO until they have reached p_1% of the total bond principal.
Tranche 2. It has p_2% of the principal and absorbs all losses in excess of p_1% of the principal up to a maximum of $q_2\% = (p_1 + p_2)\%$ of the principal....
Each Tranche has an specific yields (r_i), which represent the rates of interest paid to tranche holders. These rates are paid on the balance of the principal remaining in the tranche after losses have been paid.

n^{th} to default and m^{th} worst performances Swap

The buyer of protection (**A**) pays a specified rate on a specified notional principal until the n^{th} default occurs among a specified set of N reference entities (C_i) or until the end of the contract's life. The payments are usually made quarterly (we assume only an initial payment). If the n^{th} default occurs before the contract maturity, **A** can present bonds issued by the defaulting entity to the seller of protection (**B**) in exchange for the face value of the bonds. **A** can also presents a set of m^{th} market variables (i.e. equities, stocks) with the worst performances among a specified set of M variables, in exchange for the value of the u^{th} performance ($u > m$).

Collateralized Fund Obligation



Structural Framework

Basic assumptions.

- Firm i default as soon as $V_i(t) < D_i$. Where V_i is the firm assets value.

- $V_i(t)$ follow log-normal processes with constant drift and volatility.

- Interest rate is constant, r.

Definitions

- τ_i- time of default of firm i, then:
  $P(\tau_i < t) = P(Z_i(t) < \ln D_i)$
  $Z_i(t) = \min_{0 \le s \le t} \ln V_i(s)$.

- $\tau^j \sim$ time when j-default occurs.

- $\pi_t(j)$: The probability of exactly j default before t.

$$\pi_t(j) = P(\tau^j \le t, \tau^{j+1} > t) \tag{1}$$

$$= \sum_{i_1 \neq i_j = 1}^{\frac{k!}{j!(k-j)!}} P(Z_1(t) > D_1, ..., Z_{i_1}(t) \le D_{i_1}, ..., Z_{i_j}(t) \le D_{i_j}, ..., Z_n(t) > D_n)$$

- The probability of at least j default before t:

$$P(j,t) = P(\tau^j \leq t) \tag{2}$$
$$= [1 - P(Z_1(t) > D_1, ..., Z_i(t) > D_i, ..., Z_n(t) > D_n)]$$
$$- P(\tau^j \leq t, \tau^{j+1} > t)$$

- Denote $f_j(.)$ the density of τ^j.

Notice that the multivariate density of (τ^j, V) can be computed by using the distribution of (Z,V).

Mathematical Result

Let us assume: $X(t) = \mu \cdot t + \Sigma \cdot w(t)$, $t \geq 0$, where $\mu_{n\times 1}$ and $\sigma_{n\times n}$ are constants. Denote:

$$P(X(t) \in dx, \underline{X}(t) \geq m) = p(x, t, m, \mu, \Sigma) \prod_{i=1}^{n} dx_i$$

where $x_i > m_i \, m_i \leq 0; \forall i$.

Then p (joint density) should satisfy the Fokker-Planck PDE, with initial and absorbing boundary conditions::

$$\begin{cases} \dfrac{\partial p}{\partial t} = \sum\limits_{i=1}^{3} \mu_i \cdot \dfrac{\partial p}{\partial x_i} + \sum\limits_{i,j=1}^{n} \dfrac{\sigma_{ij}}{2} \cdot \dfrac{\partial^2 p}{\partial x_i \partial x_j} \\ p(x, t = 0) = \prod_{i=1}^{n} \delta(x_i) \\ p(x_i = m_i, t) = 0 \ , \quad i = 1, ..., n \end{cases} \tag{3}$$

Theorem 1. *The joint density can be expressed as:*

$$p(x, t, m, \mu, \Sigma) = h_1(\mu, \Sigma) \cdot h_2(x, t, m, \mu, \Sigma),$$

where h_1 *is an exponential function of* μ *and* Σ*, while* h_2 *is a linear product of Bessel, Legendre and more general Sturm-Liouville functions.*

Remark Integrating over the density functions on the above theorem, we can obtain: $P(\underline{X}_1(t) \geq m_1, , \underline{X}_N(t) \geq m_N)$.

Particular case, N=2. (See Rebholz 1998)

$$
\begin{aligned}
&P(Z_1(t) < m_1, Z_2(t) < m_2) \\
&= e^{a_1 m_1 + a_2 m_2 + bt} \cdot \frac{2}{\alpha t} \cdot \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta^*(m_1)}{\alpha}\right) \\
&\cdot e^{\frac{-g(m_1, m_2)}{2t}} \cdot \int_0^{\alpha} \sin\left(\frac{n\pi\theta(m_1)}{\alpha}\right) \cdot g_n(m_1, m_2) d\theta
\end{aligned}
\tag{4}
$$

Where $g_n(m_1, m_2)$ is an integral of Bessel functions.

Pricing n^{th} to default CDS.

Extra Assumptions

1 - Principals and the expected recovery rates associated with all the underlying entities are the same, $L = 1$, R.

2 - In the event of and an j^{th} default occurring the sellers pays the notional principal times $(1 - R)$. We could also assume $R(V_{i_1}(\tau^j), ..., V_{i_j}(\tau^j))$, we know the multivariate distribution from slide 6.

3 - We assume only one payment (from the buyer) at the beginning of the contract.

Proposition 4: The present value (t) of the expected payoff of a j^{th} to default CDS is:

$$E_t^Q \left[(1 - R) \cdot e^{-r(\tau^j - t)} \cdot 1_{\{\tau^j < T\}} \right] \tag{5}$$

$$= \int_t^T (1 - R) \cdot e^{-r(s-t)} \cdot f_j(s) ds \tag{6}$$

Pricing Percent of defaults CDO.

Extra Assumptions

1 - Principals (L) associated with all the underlying names are the same.

2 - The tranche i is responsible for between q_i% and q_{i+1}% of defaults in a CDO where there are N names.

3 - The principal to which the promised payments are applied declines as defaults occur.

Proposition 5: The present value of the expected cost of defaults for this tranche is the sum of the cost of defaults for n^{th} to default CDS for values of n between q_i% and q_{i+1}%.

Suppose that there is a promised percentage payment of r_i at time τ. In our case the payment is $p_i\% \cdot L \cdot r_i$ with probability $1 - P(q_i, \tau)$, $(p_i\% - 1\%) \cdot L \cdot r_i$ with probability $\pi_\tau(q_i)$, $(p_i\% - 2\%) \cdot L \cdot r_i$ with probability $\pi_\tau(q_i + 1)$, and so on.

The expected payment is therefore:

$$p_i \cdot r_i \cdot L \cdot [1 - P(q_i, \tau)] + \sum_{j=1}^{p_i - 1} [(p_i - j) \cdot r_i \cdot L \cdot \pi_\tau(q_i + j - 1)] \tag{7}$$

So the value today, t, for the tranche i is:

$$\int_t^T e^{-r\cdot(s-t)} \cdot (p_i \cdot r_i \cdot L)\, ds$$

$$- \int_t^T e^{-r\cdot(s-t)} \cdot (p_i \cdot r_i \cdot L) \cdot f_{q_i}(s) ds$$

$$+ \sum_{j=1}^{p_i-1} \left\{ \int_t^T e^{-r\cdot(s-t)} \cdot [(p_i - j) \cdot r_i \cdot L] \cdot f_{q_i+j-1}(s) ds \right\} \qquad (8)$$

Remark The distribution of losses $Lo(t)$ for tranche i can be obtained by noticing that $P(Lo(t) = L(t) \cdot q_i + s) = \frac{\pi_t(q_i+s)}{\sum_{l=0} \pi_t(q_i+l)}$.

n^{th} **to default and** m^{th} **worst performances Swap**:

Same Assumptions as for n^{th} to default Swap.

Proposition 4: The present value (t) of the expected payoff of a n^{th} to default and m^{th} worst performances Swap is:

$$E_t^Q[(1-R)\cdot e^{-r(\tau^j - t)}\cdot 1_{\{\tau^j<T\}}]+$$
$$E_t^Q[(m\cdot S_{i_{m+1}}(\tau^j))\cdot e^{-r(\tau^j - t)}\cdot 1_{\{\tau^j<T, S_{i_1}(\tau^j)\leq \ldots \leq S_{i_m}(\tau^j)\}}] \qquad (9)$$

Where $\{S_{i_j}(t)\}_{j=1}^{M}$ is the ordered stock prices at t (increasing).