Practical Application of Economic Credit Capital Allocation Methodologies

RiskLab Madrid
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Summary

- **Economic Credit Capital**: buffer that provides protection against credit risk faced by an institution - absorb unexpected losses (up to a level, e.g. 99.9%)
- Analytical and simulation tools available
  - Simulation - flexibility to create realistic credit models (capturing diversification, codependence, multiple asset classes and default models, stochastic modeling of exposures and LGDs)
  - Analytical methods provide insights into the behaviour and sensitivities
- In addition to measuring capital, management requires general methodologies to
  - **a posteriori**, attribute capital to sub-portfolios (e.g. activities, business units, transactions)
  - **a priori**, allocate it in an optimal fashion, to maximize risk adjusted returns.
Summary – In this talk…

• We introduce volatility, ES- and VaR-based capital allocations
  • Discuss the advantages and disadvantages, and address shortcomings of volatility for credit risk contributions

• We address perceived disadvantages of VaR-based allocations – we show that
  • They yield linear allocations of credit capital
  • Do not present some of the undesirable features of volatility-based allocations
  • Can be accurately and robustly computed in a simulation framework

• We discuss practical Issues
  • Allocation to higher quantiles (both VaR and ES) through simulation
  • Allocation to small portfolios or transactions
  • Marginal allocation to transactions on “real time” basis
Introduction: What is Capital?

Capital acts as a buffer to
• Absorb large unexpected losses
• Protect depositors and other claim holders
• Provide confidence to external investors and rating agencies on the financial health and viability of the firm

Types of capital
• Book capital
• Regulatory capital
• Economic capital

Confidence level - trade off:
• Return on capital for shareholders
• Protection to debt holders and depositors (credit rating)
Basel II IRB Approach

Regulatory Capital = \left( \sum_{j}^{n} RWA_j \right) \times 8\%

RWA = 12.5 \cdot EAD \cdot K

K = LGD \cdot \left[ N \left( \frac{N^{-1}(PD) + \sqrt{RN^{-1}(0.999)}}{\sqrt{1-R}} \right) - PD \right] \cdot MF(M, PD)

\[ KEADRWA \cdot R = 5.12 \]

Capital requirement, K = minimum capital per unit exposure

PD = obligor’s probability of default
LGD = loss given default
R = one-factor asset correlation
MF = maturity factor
M = maturity

99.9% unexpected default losses for a one-year loan

- Asymptotically fine-grained, homogeneous portfolio
- one-factor credit portfolio model
Basel II IRB Asset Correlations

\[ K = LGD \cdot \left[ N \left( \frac{N^{-1}(PD) + \sqrt{R(N^{-1}(0.999))}}{\sqrt{1-R}} \right) - PD \right] \]

- Corporate and Banking exposures
  \[ R = 0.12 \left( \frac{1 - e^{-50PD}}{1 - e^{-50}} \right) + 0.24 \left( \frac{1 - e^{-50PD}}{1 - e^{-50}} \right) \]
- Residential mortgages
  \[ R = 0.15 \]
- Revolving credit exposures
  \[ R = 0.04 \]
- Other retail credit exposures
  \[ R = 0.03 \left( \frac{1 - e^{-35PD}}{1 - e^{-35}} \right) + 0.16 \left( \frac{1 - e^{-35PD}}{1 - e^{-35}} \right) \]
Economic Capital as a Management Tool & Capital Allocation

• Economic Capital is at the center of Pillar II in Basel II: ICAAP, use test

• Powerful business management tool - a consistent metric to determine
  • Risk aggregation
  • Performance measurement
  • Asset and business allocation

• Once calculated, EC must be allocated equitably among various components (e.g., activities, business units, obligors or transactions).
  • Capital Attribution: a posteriori, attribute capital to sub-portfolios
  • Capital Allocation: a priori - maximize risk adjusted returns

• Important for
  • Management decision support and business planning
  • Performance measurement and risk based compensation
  • Pricing, profitability assessment and limit
  • Building optimal risk-return portfolios and strategies
Capital Allocation

Capital is (should be!) sub-additive → portfolio diversification

• Example: Normal portfolio loss distribution - quantile unexpected losses “proportional” to standard deviation

\[ \sigma^2(y) = \sigma^2(x_1) + \sigma^2(x_2) + 2\sigma(x_1)\sigma(x_2)\rho \]

How do we allocate capital then?

• **Stand-alone capital**: diversification benefits not passed down to the business units - each unit expected to operate on a stand-alone basis.

• **Marginal capital**: “optimal” level of group risk-taking achieved only when diversification benefits are allocated to major business units.

  • Assign to each business unit the economic capital allocation closer to its marginal contribution to the total economic capital
### Simple Example: Stand-Alone vs. Marginal Contributions

- Portfolio with 70% of one-factor capital in portfolio 1 and 30% in portfolio 2
  - Assume that diversification decreases capital to 86.3%. i.e. DF = 86.3%

<table>
<thead>
<tr>
<th>Capital One-Factor</th>
<th>% Contributions</th>
<th>Unadjusted Capital Contributions</th>
<th>Individual Factor Adjustment</th>
<th>Adjusted Capital Contributions</th>
<th>Adjusted % Contributions</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>70.0</td>
<td>70.0%</td>
<td>60.4</td>
<td>0.94</td>
<td>66.1</td>
</tr>
<tr>
<td>P2</td>
<td>30.0</td>
<td>30.0%</td>
<td>25.9</td>
<td>0.67</td>
<td>20.2</td>
</tr>
<tr>
<td>Total</td>
<td>100.0</td>
<td>100%</td>
<td>86.3</td>
<td></td>
<td>86.3</td>
</tr>
<tr>
<td>CDI</td>
<td></td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>FA</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

- Consistent with a marginal risk allocation
  - The smaller portfolio gets an adjustment factor of 67% (contributes more to the overall diversification)
  - Larger portfolio gets a 94% factor
  - Capital contributions of the portfolios are 66.1 and 20.2 (summing to 86.3)
Capital Allocation Measures

• Most common approach: marginal contributions of portfolio credit loss volatility
  • Generally ineffective since credit loss distributions are far from Normal – leads to inefficiencies and inconsistencies

• Natural choice: contributions to a VaR-based measure.
  • Conceptual shortcoming: it is not a coherent risk measure
  • Computational shortcoming: difficulties in computing accurate and stable risk contributions in simulation

• Other alternative: Expected Shortfall (ES)
  • ES is a coherent risk measure, and ES contributions can be readily computed from simulation-based credit risk models
  • Does not correspond to the standard definition of Capital,
• Additive decomposition of a portfolio risk measure, $\rho(L)$

$$\rho(L) = \sum_{i=1}^{N} C_i^\rho$$

$C_i^\rho$ is the risk contribution of obligor/position $i$

• The relative risk contribution of obligor $i$

$$R_i^\rho = \frac{C_i^\rho}{\rho(L)}$$

• If $\rho(L)$ is homogeneous of degree one

$$C_i^\rho = L_i \frac{\partial \rho}{\partial L_i}$$
Capital Allocation
Additive Risk Contributions

Additive decomposition

\[ \rho(L) = \sum_{i=1}^{N} C_i^p \]

- Volatility: \( \rho(L) \equiv \sigma(L) \)
  \[ C_i^{\sigma} = \frac{\text{cov}(L_i, L)}{\sigma(L)} \]

- VaR
  \[ C_i^{\text{VaR}_\alpha} = E(L_i \mid L = \text{VaR}_\alpha(L)) \]

- Expected Shortfall
  \[ C_i^{\text{ES}_\alpha} = E(L_i \mid L \geq \text{VaR}_\alpha(L)) \]
Simulation-Based Approach

### e.g. 100 scenarios

- **VaR(95%)** = 5th largest loss
- **ES(95%)** = average of 5 largest losses

<table>
<thead>
<tr>
<th>Securities</th>
<th>Positions</th>
<th>Portfolio</th>
<th>Empirical distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 2 8 4</td>
<td>100</td>
<td>1280</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>200</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Mark-to-Future table (fixed)
VaR (Quantile) Estimation

• if $F$ is known, it’s easy to calculate VaR
  
  • e.g., if $L \sim N(0, \sigma)$
    
    $$VaR(\alpha) = Z_\alpha \sigma$$
  
  where $Z_\alpha$ is the $\alpha$th quantile of the standard normal distribution

• if $F$ is unknown, then VaR has to be estimated from the sample data
  
  • If $Q_\alpha$ is an estimate of the $\alpha$th quantile of the unknown loss distribution then
    
    $$VaR(\alpha) = Q_\alpha$$

• Non-parametric approach makes no distributional assumptions
Non-Parametric VaR

e.g., 100 scenarios, each with probability 0.01; find VaR(95%)

- Given a set of losses \( \{ L_i : i = 1, 2, \ldots, 100 \} \)

<table>
<thead>
<tr>
<th>560</th>
<th>147</th>
<th>-25</th>
<th>893</th>
<th>14</th>
<th>-74</th>
<th>...</th>
<th>293</th>
<th>62</th>
<th>581</th>
<th>-295</th>
<th>-672</th>
<th>356</th>
</tr>
</thead>
</table>

- Obtain order statistics (sort losses in increasing sequence)

```
1  2  3  4  5  6  95  96  97  98  99  100
-946  -815  -672  -617  -453  -311  ...  581  652  724  893  927  1112
```

- Starting from the largest loss, accumulate probability until reaching 0.05
- \( \text{VaR}(95\%) = 652 \) is the 5\(^{th}\) largest loss (i.e., the 96\(^{th}\) order statistic, \( L_{(96)} \))

- Easy to get non-parametric bounds on VaR, as well
Another VaR Estimator

e.g., 100 scenarios, each with probability 0.01; find VaR(95%)

• Given a set of losses \{L_i : i = 1, 2, \ldots, 100\}

\[
\begin{array}{cccccccc}
\end{array}
\]

• Obtain order statistics (sort losses in increasing sequence)

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>95</th>
<th>96</th>
<th>97</th>
<th>98</th>
<th>99</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>-946</td>
<td>-815</td>
<td>-672</td>
<td>-617</td>
<td>-453</td>
<td>-311</td>
<td>\ldots</td>
<td>581</td>
<td>652</td>
<td>724</td>
<td>893</td>
<td>927</td>
</tr>
</tbody>
</table>

• Starting from the smallest loss, accumulate probability until reaching 0.95

• \text{VaR}(95%) = 581 is the 6^{th} largest loss (i.e., the 95^{th} order statistic, \(L_{(95)}\))
Multiple Order Statistics

• Both the UECV and inverse empirical cdf are based on a single order statistic
• Maybe “nearby” order statistics also represent useful information for estimating quantiles?
• Consider taking a weighted average of multiple order statistics
• Can derive weights based on:
  • Smoothing the empirical cdf
  • Statistical arguments (e.g., bootstrapping)
L-estimators

- More generally, the quantile estimate can be a weighted average of all the order statistics

\[ Q_\alpha = \sum_{k=1}^{S} w_{\alpha,S,k} L_{(k)} \quad \text{where} \quad \sum_{k=1}^{S} w_{\alpha,S,k} = 1 \]

- These are known as “L-estimators”

- For the inverse empirical cdf

\[ w_{\alpha,S,k} = \begin{cases} 1 & \text{if } k = \lfloor S\alpha + 1 \rfloor - 1 \\ 0 & \text{otherwise} \end{cases} \]

- For the UECV

\[ w_{\alpha,S,k} = \begin{cases} 1 & \text{if } k = \lfloor S\alpha \rfloor + 1 \\ 0 & \text{otherwise} \end{cases} \]
Harrell-Davis Estimator

• if $F$ is known, then the expected value of the $k^{th}$ order statistic is

$$E(L_{(k)}) = \int_{-\infty}^{\infty} S \left(\frac{S-1}{k-1}\right) F(x)^{k-1} \left[1 - F(x)\right]^{S-k} f(x) \cdot x \, dx$$

$$= \frac{1}{\beta\{k, S-k+1\}} \int_{0}^{1} F^{-1}(y) y^{k-1} (1 - y)^{S-k} \, dy$$

• if $F$ is unknown, just replace with $\hat{F}(y)$

• for $0 < \alpha < 1$, $E(L_{((S+1)\alpha)}) \to F^{-1}(\alpha)$ as $S \to \infty$, so...

$$Q_\alpha = \frac{1}{\beta\{(S+1)\alpha, (S+1)(1-\alpha)\}} \int_{0}^{1} \hat{F}^{-1}(y) y^{(S+1)\alpha-1} (1 - y)^{(S+1)(1-\alpha)-1} \, dy$$

$$= L_{(k)} \text{ for } \frac{k-1}{S} < y \leq \frac{k}{S}$$

• this is an L-estimator: $Q_\alpha = \sum_{k=1}^{S} w_{\alpha,S,k} L_{(k)}$ where

$$w_{\alpha,S,k} = \frac{1}{\beta\{(S+1)\alpha, (S+1)(1-\alpha)\}} \int_{(k-1)/S}^{k/S} y^{(S+1)\alpha-1} (1 - y)^{(S+1)(1-\alpha)-1} \, dy$$
Example Harrell-Davis Weights

$S = 100$

$\alpha = 0.95$
Use of L-estimators

- Empirically, Harrell-Davis (HD) has been found to work well
  - In practice, the sample quantile (UECV) is often “good enough” - modest improvement with sophistication (Sheather & Marron (1990) ~15% better MSE)
- In general, results (e.g., bias, efficiency) depend on quantile & sample size
- Need other tools to get high quantiles
  - With L-estimators, estimated quantile cannot exceed the largest order statistic
  - Extreme Value Theory, Importance Sampling are useful tools...
- L-estimators become important tools when we estimate quantile sensitivities (e.g., marginal VaR, risk contributions)
  - Differences between UECV and HD are significant!
L-estimators & Piecewise Linearity of VaR

• Let $L_{(k)}(x)$ be the $k^{th}$ smallest loss for a portfolio containing positions $x$

• Portfolio VaR, as given by any L-estimator is

$$VaR(\alpha; x) = \sum_{k=1}^{S} w_{\alpha,S,k} L_{(k)}(x)$$

• Let $\Delta v_{i(k)}$ be the per-unit loss of instrument $i$ in the scenario that results in $k^{th}$ smallest loss, given positions $x$

• Portfolio VaR is a (piecewise) linear function of the position sizes:

$$VaR(\alpha; x) = \sum_{k=1}^{S} w_{\alpha,S,k} \sum_{i=1}^{N} \Delta v_{i(k)} x_i$$

$$= \sum_{i=1}^{N} \left\{ \sum_{k=1}^{S} w_{\alpha,S,k} \Delta v_{i(k)} \right\} x_i$$

$$= \sum_{i=1}^{N} \omega_{\alpha,S,i} x_i$$

(weighted per-unit loss (constant as long as the order of the scenarios, ranked by portfolio loss, does not change))
Simple Example 1: VaR (95%)

\[
V a R(\alpha; x) = \sum_{i=1}^{N} \omega_{\alpha,S,i} x_i
\]

<table>
<thead>
<tr>
<th>rank</th>
<th>( \Delta v )</th>
<th>x</th>
<th>L(x)</th>
</tr>
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<tbody>
<tr>
<td>100</td>
<td>1.32 0.90</td>
<td>A B</td>
<td>Portf.</td>
</tr>
<tr>
<td>99</td>
<td>1.25 1.00</td>
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<td>1770</td>
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<tr>
<td>98</td>
<td>1.17 0.70</td>
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<td>1750</td>
</tr>
<tr>
<td>97</td>
<td>0.90 0.75</td>
<td>A B</td>
<td>1520</td>
</tr>
<tr>
<td>96</td>
<td>0.83 0.60</td>
<td>A B</td>
<td>1275</td>
</tr>
<tr>
<td>95</td>
<td>0.64 0.80</td>
<td>A B</td>
<td>1130</td>
</tr>
<tr>
<td>94</td>
<td>0.71 0.58</td>
<td>A B</td>
<td>1040</td>
</tr>
<tr>
<td>93</td>
<td>0.60 0.46</td>
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</tr>
<tr>
<td>92</td>
<td>0.65 0.35</td>
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<td>830</td>
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<tr>
<td>91</td>
<td>0.58 0.36</td>
<td>A B</td>
<td>825</td>
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<tr>
<td>90</td>
<td>0.47 0.38</td>
<td>A B</td>
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</tr>
<tr>
<td>89</td>
<td>0.50 0.32</td>
<td>A B</td>
<td>660</td>
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<table>
<thead>
<tr>
<th>( \omega_{0.95,100,A} )</th>
<th>( \omega_{0.95,100,B} )</th>
<th>VaR(95%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UECV</td>
<td>0.83 0.60</td>
<td>1130</td>
</tr>
<tr>
<td>HD</td>
<td>0.82 0.65</td>
<td>1147</td>
</tr>
</tbody>
</table>
**VaR Sensitivities & L-Estimators**

\[ VaR(\alpha; x) = \sum_{i=1}^{N} \omega_{\alpha,S,i} x_i \]

- Marginal VaR of instrument \( i \) equals its weighted per-unit loss

\[ \frac{\partial VaR(\alpha; x)}{\partial x_i} = \omega_{\alpha,S,i} \]

- Marginal decomposition - the risk contribution of instrument \( i \) is

\[ C(x_i) = \frac{\omega_{\alpha,S,i} x_i}{VaR(\alpha; x)} \times 100\% \]
## Simple Example 2: VaR (95%)

### VaR Calculation

\[
    \text{VaR}(\alpha; x) = \sum_{i=1}^{N} \omega_{x,\alpha,i} x_i
\]

### Marginal VaR

What is the correct marginal VaR?

<table>
<thead>
<tr>
<th>rank</th>
<th>( \Delta m )</th>
<th>( x )</th>
<th>( L(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1.32</td>
<td>1000</td>
<td>870</td>
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<td>99</td>
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</tr>
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<td>97</td>
<td>0.75</td>
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</tr>
<tr>
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<td>550</td>
</tr>
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</tr>
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<td>1000</td>
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</tr>
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<td>1000</td>
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<td>-0.02</td>
<td>1000</td>
<td>355</td>
</tr>
<tr>
<td>89</td>
<td>-0.10</td>
<td>1000</td>
<td>315</td>
</tr>
</tbody>
</table>
Some Theoretical Results on Marginal Risk

- Implicitly, we use the general theorem (c.f. Tasche (1999) or Gourieoux et al (2000))

\[
\frac{\partial VaR(\alpha; x)}{\partial x_i} = E[\Delta v_i \mid \Delta v = VaR(\alpha; x)]
\]

- Thus, assuming that the empirical loss distribution from the simulation is the actual loss distribution, this results in the expression given previously

\[
\frac{\partial VaR(\alpha; x)}{\partial x_i} = \Delta v_{i,s^o}
\]

(simply take losses of the i-th position in the percentile scenario (average if there are more than one)

- Intuition behind L-estimators: weights of estimator can be interpreted as the probabilities of those scenarios conditional on the given VaR

- Laurent (2003) presents further theoretical work

\[
\frac{\partial VaR(\alpha; x)}{\partial x_i} = \omega_{\alpha,s,i} = \sum_{k=1}^{S} w_{\alpha,s,k} \Delta v_{i(k)}
\]
Credit Risk Example
Emerging Markets Bond Portfolio

bond portfolio

• 86 obligors in 29 countries

M tM = 8.3 billion USD

<table>
<thead>
<tr>
<th>Bond Type</th>
<th>Market Value</th>
<th>% Market Value</th>
<th>No. of Instruments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brady Bonds</td>
<td>3,581</td>
<td>40.7%</td>
<td>54</td>
</tr>
<tr>
<td>Fixed Rate Bonds</td>
<td>5,178</td>
<td>58.9%</td>
<td>140</td>
</tr>
<tr>
<td>Floating Rate Notes</td>
<td>30</td>
<td>0.3%</td>
<td>3</td>
</tr>
<tr>
<td><strong>TOTAL</strong></td>
<td><strong>8,789</strong></td>
<td><strong>62.9%</strong></td>
<td><strong>197</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>% Number of Instruments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sovereign</td>
</tr>
<tr>
<td>State Owned</td>
</tr>
<tr>
<td>Municipalities</td>
</tr>
<tr>
<td>Corporations</td>
</tr>
<tr>
<td>Financial</td>
</tr>
<tr>
<td>Brady Bonds</td>
</tr>
<tr>
<td>26.4%</td>
</tr>
<tr>
<td>1.0%</td>
</tr>
<tr>
<td>0.0%</td>
</tr>
<tr>
<td>0.0%</td>
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<tr>
<td>27.4%</td>
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<td>18.8%</td>
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<td>0.0%</td>
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<tr>
<td>0.0%</td>
</tr>
<tr>
<td>0.0%</td>
</tr>
<tr>
<td>0.5%</td>
</tr>
<tr>
<td>1.5%</td>
</tr>
</tbody>
</table>

Simulated credit losses for one-year horizon
(20,000 scenarios)

VaR(99%)
UECV: 1.026 B USD
HD: 1.028 B USD
Credit Simulation Methodology

• Compute exposures and calculate credit losses for each obligor in each credit state

<table>
<thead>
<tr>
<th></th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Obligor 1</td>
<td>0</td>
<td>5</td>
<td>11</td>
<td>15</td>
<td>22</td>
<td>35</td>
<td>53</td>
<td>100</td>
</tr>
<tr>
<td>Obligor 2</td>
<td>-15</td>
<td>-7</td>
<td>0</td>
<td>10</td>
<td>16</td>
<td>30</td>
<td>48</td>
<td>135</td>
</tr>
<tr>
<td>Obligor 3</td>
<td>-5</td>
<td>0</td>
<td>4</td>
<td>11</td>
<td>17</td>
<td>26</td>
<td>39</td>
<td>75</td>
</tr>
</tbody>
</table>

• Sample the joint credit states of obligors to obtain portfolio credit losses

Scen 1: 0 + 0 + 0 = 0
Scen 2: 0 + 10 + 0 = 10
Scen 3: 5 + (-7) + 0 = -2

• An obligor can have one of a only small set of possible losses
  • Usually loss is zero (no change in credit state)
  • Potential problems with UECV sensitivity estimates
Risk Contributions: Obligor contributions to portfolio credit risk

- Obligors ranked by contributions to $\sigma$ (or unexpected loss)
- 5 obligors > 50% of portfolio credit risk!
- Note: expected losses are non-diversifiable

<table>
<thead>
<tr>
<th>Rating</th>
<th>Mark to Market</th>
<th>% Contribution to Portfolio</th>
<th>Std Deviation</th>
<th>Loss at 99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brazil</td>
<td>BB 880</td>
<td>14.55%</td>
<td>17.10%</td>
<td>20.27%</td>
</tr>
<tr>
<td>Venezuela</td>
<td>B 398</td>
<td>6.16%</td>
<td>14.10%</td>
<td>12.25%</td>
</tr>
<tr>
<td>Russia</td>
<td>BB 756</td>
<td>9.81%</td>
<td>12.21%</td>
<td>14.31%</td>
</tr>
<tr>
<td>Argentina</td>
<td>BB 624</td>
<td>9.87%</td>
<td>9.33%</td>
<td>10.47%</td>
</tr>
<tr>
<td>Peru</td>
<td>BB 283</td>
<td>10.33%</td>
<td>9.00%</td>
<td>8.30%</td>
</tr>
<tr>
<td>Colombia</td>
<td>BBB 605</td>
<td>2.30%</td>
<td>2.97%</td>
<td>3.26%</td>
</tr>
<tr>
<td>Russia</td>
<td>CC 48</td>
<td>1.29%</td>
<td>2.51%</td>
<td>1.80%</td>
</tr>
<tr>
<td>Mexico</td>
<td>BB 488</td>
<td>9.20%</td>
<td>1.96%</td>
<td>1.69%</td>
</tr>
<tr>
<td>Morocco</td>
<td>BB 124</td>
<td>1.58%</td>
<td>1.36%</td>
<td>0.82%</td>
</tr>
<tr>
<td>Philippines</td>
<td>BB 448</td>
<td>6.67%</td>
<td>1.22%</td>
<td>0.26%</td>
</tr>
<tr>
<td>Panama</td>
<td>BB 74</td>
<td>1.57%</td>
<td>1.21%</td>
<td>0.83%</td>
</tr>
<tr>
<td>Bulgaria</td>
<td>B 198</td>
<td>0.16%</td>
<td>1.20%</td>
<td>0.16%</td>
</tr>
<tr>
<td>Turkey</td>
<td>B 114</td>
<td>3.98%</td>
<td>1.14%</td>
<td>0.89%</td>
</tr>
<tr>
<td>Romania</td>
<td>BB 87</td>
<td>0.78%</td>
<td>0.87%</td>
<td>0.16%</td>
</tr>
</tbody>
</table>
Marginal Portfolio Credit Risk

Risk contribution ~ Marginal risk (additional risk/unit) X size of exposure

Outliers:
- Russia CCC
- Peru
- Venezuela
- Argentina
- Russia
- Brazil
- Angola
- China
- Colombia
- Mexico
- Poland

Mean exposure at one year (millions USD)
## Largest Contributors to Risk

<table>
<thead>
<tr>
<th></th>
<th>Value (MUSD)</th>
<th>Contribution (%)</th>
<th>$1 Marginal VaR 99%</th>
<th>Best Hedge Position (MUSD)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>UECV</td>
<td>UECV-s</td>
<td>HD</td>
</tr>
<tr>
<td>Brazil</td>
<td>894</td>
<td>24.69</td>
<td>24.62</td>
<td>25.85</td>
</tr>
<tr>
<td>Russia</td>
<td>758</td>
<td>15.12</td>
<td>17.23</td>
<td>19.22</td>
</tr>
<tr>
<td>Venezuela</td>
<td>414</td>
<td>7.23</td>
<td>13.96</td>
<td>13.30</td>
</tr>
<tr>
<td>Argentina</td>
<td>636</td>
<td>8.83</td>
<td>11.55</td>
<td>12.47</td>
</tr>
<tr>
<td>Peru</td>
<td>279</td>
<td>17.54</td>
<td>7.98</td>
<td>7.57</td>
</tr>
<tr>
<td>Colombia</td>
<td>608</td>
<td>7.12</td>
<td>2.36</td>
<td>2.35</td>
</tr>
<tr>
<td>Mexico</td>
<td>491</td>
<td>0.00</td>
<td>1.70</td>
<td>2.09</td>
</tr>
<tr>
<td>Russia CCC</td>
<td>44</td>
<td>2.02</td>
<td>1.70</td>
<td>1.90</td>
</tr>
</tbody>
</table>

**risk contribution not detected**
Efficiency w.r.t. Marginal VaR

- Estimate marginal VaR for Brazil by sub-sampling the 20,000 scenarios (without replacement)
- Consider 1,000 sub-samples of various sizes

<table>
<thead>
<tr>
<th>Sample size</th>
<th>UECV</th>
<th>HD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,000</td>
<td>0.28 (0.25)</td>
<td>0.29 (0.08)</td>
</tr>
<tr>
<td>5,000</td>
<td>0.31 (0.26)</td>
<td>0.29 (0.04)</td>
</tr>
<tr>
<td>10,000</td>
<td>0.29 (0.25)</td>
<td>0.30 (0.03)</td>
</tr>
</tbody>
</table>

Increasing the sample size doesn’t help (lacks consistency)!

Sample Size = 10,000

20% of the time we would conclude that Brazil does not contribute to risk
VaR and ES for a range of quantile levels

- UECV and HD estimators yield virtually identical results
Risk Contributions of Brazilian Debt

- UECV and HD estimators give similar results for ES but not for VaR.
- The HD estimator consistently identifies Brazil as a significant source of risk under both measures.
- UECV VaR contributions are erratic and frequently fail to attribute risk.
Scenarios in the Tail: Largest Portfolio Losses in Sample

- Figure: tail of the empirical portfolio loss distribution beyond the 98% and the component losses due to Brazilian bonds.

- Portfolio loss profile is relatively smooth
- Brazilian losses often change drastically
  - Obligor incurs only one of eight possible losses (eight possible credit states in each scenario.)
Convergence of Estimators

• VaR: in practice, for a sufficiently large sample, single-order statistic quantile estimator (UECV) produces robust VaR estimates and improve with sample size

• In contrast, UECV estimates of VaR contributions are unreliable
  • Since smoothness of the obligor loss profile is unaffected by sample size, accuracy does not necessarily improve with the number of scenarios.

• Significant implications for credit risk models
  • Discrete losses triggered by obligor’s default or transition to a lower credit
  • High probability of an individual obligor retaining original credit rating → many obligors do not incur a loss in a given scenario
  • UECV estimator results in excessive number of zero VaR contributions.
What Risk Contribution for Capital Allocation?

- ES and VaR contributions depend on the quantile chosen.

- Volatility allocation is problematic - may fail to represent the tail of the distribution.

- Tail-based contributions consistently exceed volatility contribution for two largest contributors, (Brazil and Russia).

- Rankings of Russia and Venezuela are reversed when based on volatility.

Tail-based (VaR and ES) risk contributions - quantile levels between 98% and 99.9%, against volatility contributions.
Allocation to High Quantiles in Simulation

- Whether using VaR or ES, it is difficult (and expensive) to estimate accurately credit capital and risk contributions for high quantiles (e.g. 99.9% or 99.97%) in simulation
  - Not enough scenarios in the tail to calculate (conditional) expectations

- Practical solutions:
  - Compute capital (and risk contributions) for more reasonable quantiles and scale based on analytical model
  - Use of advanced MC (variance reduction) tools
    - Importance sampling
    - Control variates
Allocation to Small Portfolios or Transactions

- How can risk contributions be measured accurately when contributions are very small (e.g. for very large portfolios) – contribution size way beyond the accuracy of the estimator (and any parameters used in the model)?
  - Examples: retail portfolio with millions of transactions, large corporate or enterprise portfolios

- Practical solutions:
  - Application of a simpler (calibrated) analytical model
  - Use of a consistent hierarchical methodology to allocate contributions through a large portfolio – e.g. use of basic properties of granular (homogeneous) portfolios
Application of Simple Models to “Real Time” Capital Contributions

• How can marginal capital be computed to a new loan or transaction (accounting properly for diversifications? (in this case also contributions are very small)

• Practical solutions:
  • Full simulation in this case might not work (same problem as before) - some semi-analytical simulation models might work
  • Application of a simpler analytical model - calibrated to full economic capital model
  • Important to be intuitive and use a small number of parameters
  • Pure statistical parameterization (recalibrated frequently over time)
Summary - In this talk...

- We introduce volatility, ES- and VaR-based capital allocations
  - Discuss the advantages and disadvantages, and address shortcomings of volatility for credit risk contributions
- We address perceived disadvantages of VaR-based allocations - we show that
  - They yield linear allocations of credit capital
  - Do not present some of the undesirable features of volatility-based allocations
  - Can be accurately and robustly computed in a simulation framework
- We discuss practical Issues
  - Allocation to higher quantiles (both VaR and ES) through simulation
  - Allocation to small portfolios or transactions
  - Marginal allocation to transactions on “real time” basis