A New Model for the Pricing of Defaultable Bonds

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A New Model for the Pricing of Defaultable Bonds

Overview

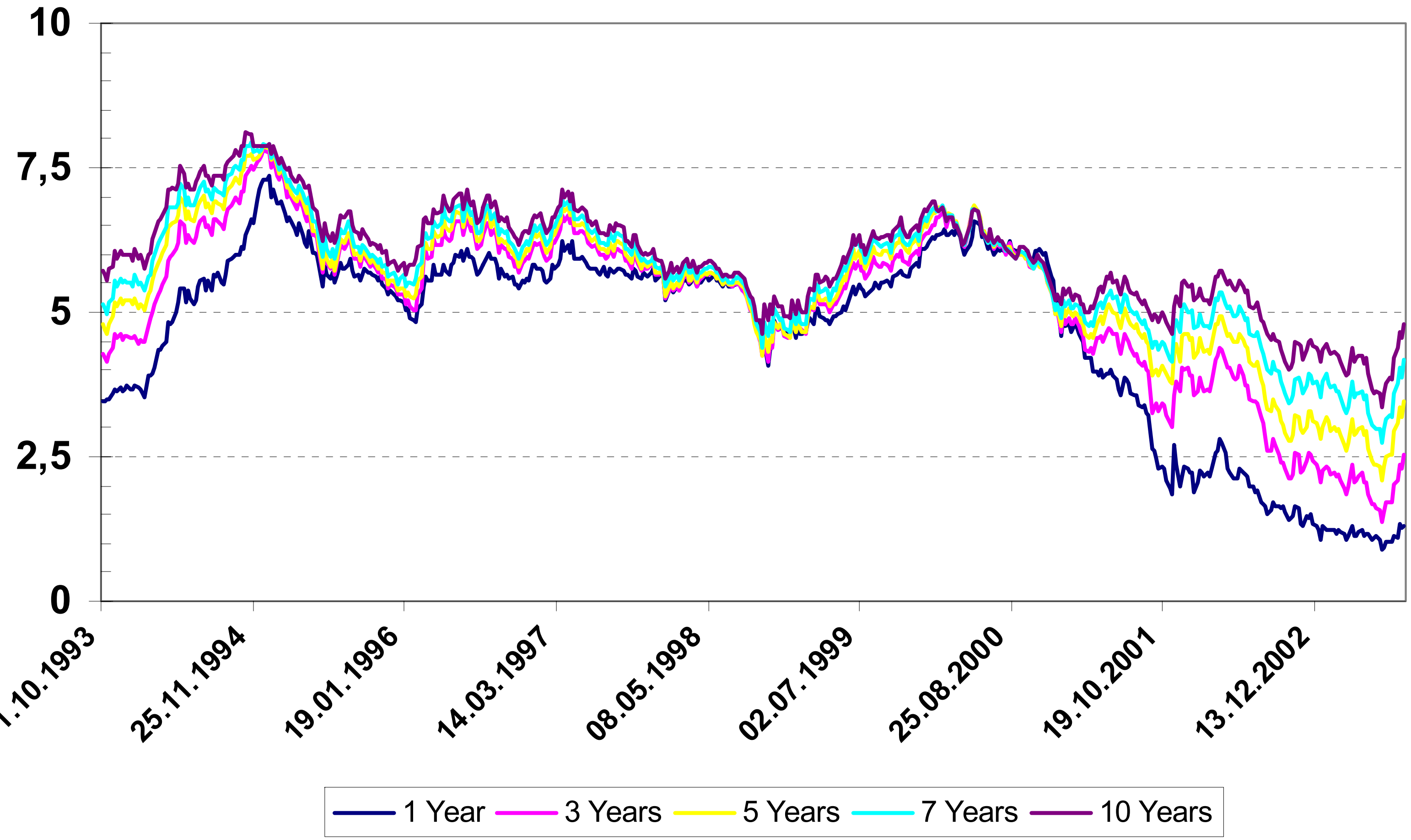
- **Market Information**
  - Yield Curve Behaviour
    US Treasury Strips
  - Credit Spread Behaviour
    US Industrials A2
  - Credit Spread Behaviour
    US Industrials BBB1
  - Economic Behaviour
    US Gross Domestic Product (GDP)
- **Three Models for the Pricing of Defaultable Bonds**
- **Model Comparison**
- **Further Research**

Treasury and Spread Data
US Market 1993 - 2003

- American Treasury Strips
- A2- and BBB1-rated American Industrials
- Maturities between 3 months and 30 years
- Time series of weekly bond prices from Oktober 8, 1993 to June 1, 2001 (in sample)
- Time series of weekly bond prices from June 8, 2001 to August 15, 2003 (out of sample)
- All prices in US Dollar, i.e. no currency risk in credit spreads
- Parameter estimation using Kalman filters

1- to 10-Year US Treasury Strips in %

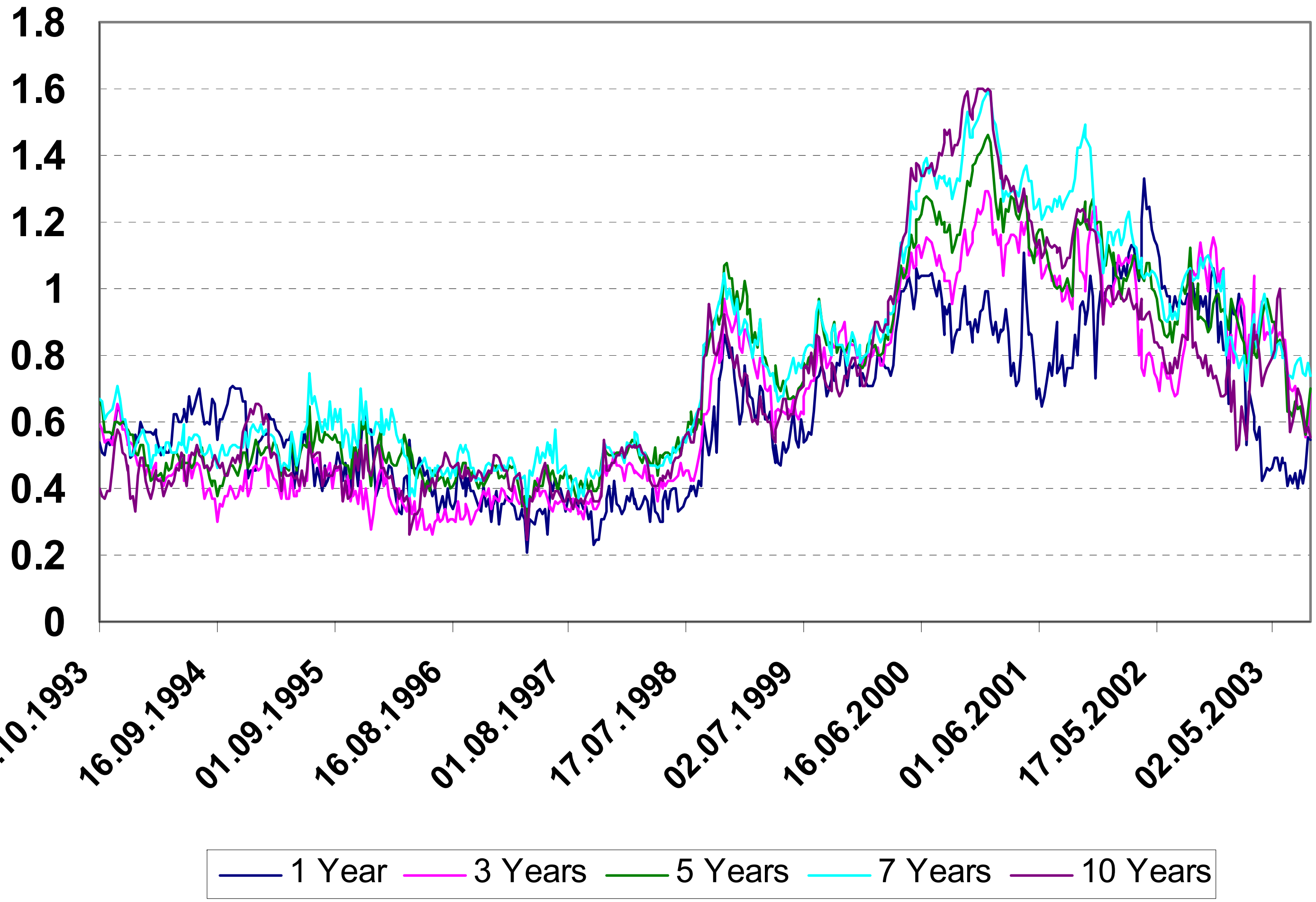
Time Period: 1993 – 2003



Source: Bloomberg

1- To 10-Year Credit Spread of US Industrials A2 in %

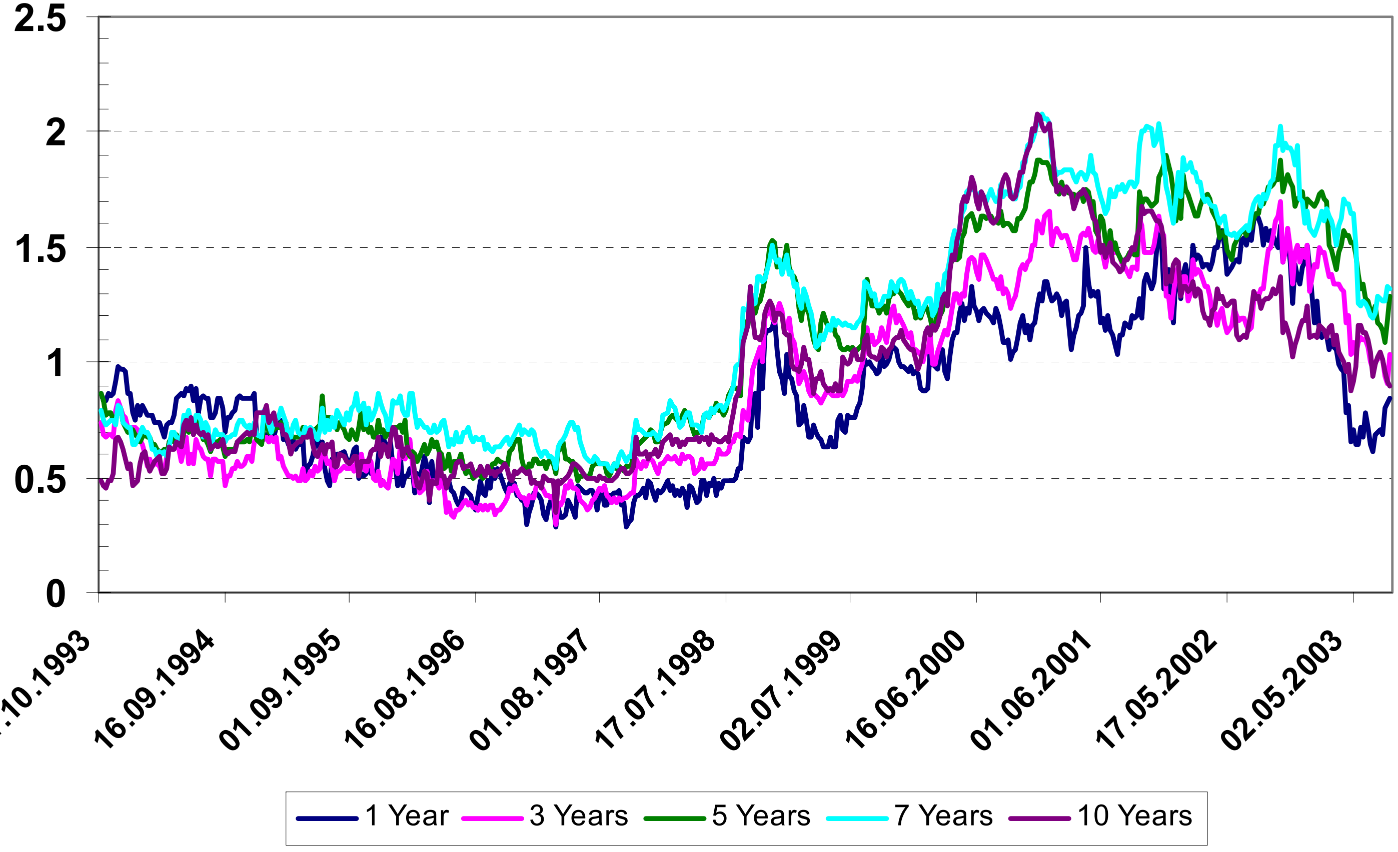
Time Period: 1993 – 2003



Source: Bloomberg

1- To 10-Year Credit Spread of US Industrials BBB1 in %

Time Period: 1993 – 2003



Source: Bloomberg

A New Model for the Pricing of Defaultable Bonds
Overview

- **Market Information**
- **Three Models for the Pricing of Defaultable Bonds**
  - **Schmid and Zagst [2000]**
    (Spread depends on firm-specific factor)
  - Bakshi, Madan, Zhang [2001]
    (Spread depends on Treasury rates & firm-specific factor)
  - Generalized Schmid and Zagst [2004]
    (Spread depends on macroeconomic & firm-specific factor)
- **Model Comparison**
- **Further Research**

The Model of Schmid and Zagst [2000]

Modeling of the Stochastic Processes Under the Martingale Measure Q

- Dynamics of the yield curve (non-defaultable short rate)

$$dr(t) = [\theta_r(t) - a_r \cdot r(t)]dt + \sigma_r dW_r(t), t \in [0, T^*], a_r > 0, \sigma_r > 0$$

- Dynamics of the unobservable counterparty quality (uncertainty index)*

$$du(t) = [\theta_u - a_u \cdot u(t)]dt + \sigma_u \sqrt{u(t)} dW_u(t), t \in [0, T^*], \theta_u \geq 0, a_u > 0, \sigma_u > 0$$

- Dynamics of the yield spreads (short-rate credit spreads)

$$ds(t) = [b_s \cdot u(t) - a_s \cdot s(t)]dt + \sigma_s \sqrt{s(t)} dW_s(t), t \in [0, T^*], b_s > 0, a_s > 0, \sigma_s > 0$$

- The Wiener processes W_r, W_u, and W_s are uncorrelated

* Leverage and book-to-market can be described by mean-reverting stochastic processes (see, e.g. Bakshi, Madan, Zhang [2001])

The Model of Schmid and Zagst [2000]
Pricing of Non-Defaultable Bonds

Theorem 1 (Hull and White [1990] , Hull[1997]).
The time t value P(r,t,T) of a non-defaultable zero-coupon bond with maturity T≥t is given by

$$P(r,t,T) = e^{A(t,T) - B(t,T)\cdot r}$$

with

$$B(t,T) = \frac{1}{a_r} \cdot \left(1 - e^{-a_r \cdot (T-t)}\right)$$

and

$$A(t,T) = \ln\left(\frac{P(r,0,T)}{P(r,0,t)}\right) - B(t,T) \cdot \frac{\partial \ln P(r,0,t)}{\partial t} - \frac{\sigma_r^2}{4a_r^3} \cdot \left(e^{-a_r \cdot T} - e^{-a_r \cdot t}\right)^2 \cdot \left(e^{2\cdot a_r \cdot t} - 1\right).$$

The Model of Schmid and Zagst [2000]

Pricing of Defaultable Bonds

Theorem 2 (Schmid and Zagst [2000]).
The time t value $P^d(r,s,u,t,T)$ of a defaultable zero-coupon bond with maturity $T \geq t$ is given by

$$\begin{aligned} P^d(r,s,u,t,T) &= e^{A^d(t,T) - B(t,T)\cdot r - C(t,T)\cdot s - D(t,T)\cdot u} \\ &= P(r,t,T)\cdot e^{A^*(t,T) - C(t,T)\cdot s - D(t,T)\cdot u} \end{aligned}$$

where

systematic

firm specific

$$C(t,T) = \frac{1 - e^{-\delta_s\cdot(T-t)}}{K_s^1 - K_s^2 \cdot e^{-\delta_s\cdot(T-t)}} \quad \text{with} \quad \delta_s = \sqrt{a_s^2 + 2\cdot\sigma_s^2}$$

$$\text{and} \quad K_s^k = \frac{1}{2}\cdot\left(a_s - (-1)^k \cdot \delta_s\right), k \in \{1,2\}$$

$$D(t,T) = -\frac{2\cdot v'(t,T)}{\sigma_u^2 \cdot v(t,T)} \quad \text{with} \quad v(t,T) \text{ complicated}$$

$$A^d(t,T) = A(t,T) + A^*(t,T) \quad \text{with} \quad A^*(t,T) = \frac{2\cdot\theta_u}{\sigma_u^2}\cdot\ln\left|\frac{v(T,T)}{v(t,T)}\right|$$

The Model of Schmid and Zagst [2000]

Parameter Estimation

Parameter	Estimation Treasury Strips
a_r ($a_{r,real}$)	0.03672 (0.11442)
σ_r (%)	0.83004
real-world mean reversion level for r (%)	6.11

Average observed 3-Month Rates:

Treasury:	5.12 %
Spreads A2:	0.59 %
Spreads BBB1:	0.76 %

Parameter	Estimation A2	Estimation BBB1
a_s ($a_{s,real}$)	0.94569 (1.19995)	0.96698 (1.34760)
σ_s (%)	0.50424	6.16900
real-world mean reversion level for s (%)	0.41	0.63
θ_u (%)	0.1886	0.03147
a_u ($a_{u,real}$)	0.19613 (0.19990)	0.00710 (0.04683)
σ_u (%)	0.11206	3.63922
real-world mean reversion level for u (%)	0.94	0.67

The Model of Schmid and Zagst [2000]

Linear Regression of Model vs. Empirical Treasury Strips*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-1.65553*10^{-5}$ ($1.86057*10^{-6}$)	0.81293 (0.81693)	0.6985 (0.7500)
3 Years	$-9.28756*10^{-6}$ ($-4.28053*10^{-6}$)	1.02622 (1.03008)	0.9442 (0.9560)
5 Years	$-1.34277*10^{-6}$ ($-2.68422*10^{-6}$)	1.09104 (1.08467)	0.9872 (0.9926)
7 Years	$4.73147*10^{-6}$ ($1.03906*10^{-6}$)	1.09526 (1.08400)	0.9513 (0.9606)
10 Years	$6.51812*10^{-6}$ ($-1.95064*10^{-6}$)	1.08763 (1.07958)	0.9211 (0.9404)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$-1.25500*10^{-4}$ ($-7.80711*10^{-5}$)	0.71893 (0.79067)	0.6162 (0.7060)
3 Years	$-4.89500*10^{-5}$ ($-2.99150*10^{-5}$)	1.03726 (0.99671)	0.9139 (0.9294)
5 Years	$1.03988*10^{-5}$ ($1.96055*10^{-5}$)	1.09909 (1.08601)	0.9917 (0.9921)
7 Years	$3.90855*10^{-5}$ ($4.39230*10^{-5}$)	1.06966 (1.09057)	0.9633 (0.9729)
10 Years	$6.27278*10^{-5}$ ($4.73641*10^{-5}$)	1.03269 (1.07702)	0.9180 (0.9482)

* Values in brackets are without 2.5% of max. outliers

The Model of Schmid and Zagst [2000]

Linear Regression of A2 - Model vs. Empirical Credit Spreads*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-4.40045 \cdot 10^{-6}$ ($-1.00039 \cdot 10^{-5}$)	1.49338 (1.38773)	0.6571 (0.6690)
3 Years	$-4.34480 \cdot 10^{-6}$ ($5.21923 \cdot 10^{-6}$)	1.43305 (1.36455)	0.4523 (0.4622)
5 Years	$-8.60220 \cdot 10^{-6}$ ($-2.17228 \cdot 10^{-6}$)	1.68180 (1.53227)	0.3904 (0.3816)
7 Years	$-6.22185 \cdot 10^{-6}$ ($-3.26028 \cdot 10^{-6}$)	1.77772 (1.55750)	0.2930 (0.2745)
10 Years	$2.34010 \cdot 10^{-6}$ ($-1.89566 \cdot 10^{-6}$)	1.64530 (1.51296)	0.1682 (0.1680)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$2.64323 \cdot 10^{-6}$ ($-9.38798 \cdot 10^{-6}$)	1.48722 (1.35270)	0.6571 (0.6411)
3 Years	$2.72865 \cdot 10^{-5}$ ($5.33658 \cdot 10^{-6}$)	1.72891 (1.46988)	0.3975 (0.3304)
5 Years	$1.34177 \cdot 10^{-5}$ ($1.86122 \cdot 10^{-5}$)	1.49348 (1.28500)	0.4404 (0.4103)
7 Years	$-1.27622 \cdot 10^{-5}$ ($2.28318 \cdot 10^{-5}$)	0.80991 (0.69241)	0.1215 (0.1289)
10 Years	$-2.28218 \cdot 10^{-5}$ ($-5.22126 \cdot 10^{-6}$)	0.80043 (0.65427)	0.0437 (0.0406)

* Values in brackets are without 2.5% of max. outliers

The Model of Schmid and Zagst [2000]

Linear Regression of BBB1 - Model vs. Empirical Credit Spreads*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-5.14297*10^{-6}$ ($-7.51586*10^{-6}$)	1.50472 (1.41491)	0.8622 (0.8503)
3 Years	$-3.06176*10^{-6}$ ($6.50870*10^{-6}$)	1.14942 (1.11749)	0.5321 (0.5643)
5 Years	$-9.55252*10^{-7}$ ($3.34430*10^{-6}$)	0.87634 (0.77505)	0.2893 (0.2968)
7 Years	$-1.34109*10^{-7}$ ($-9.50104*10^{-6}$)	0.94840 (0.92787)	0.3315 (0.3489)
10 Years	$-7.31040*10^{-7}$ ($1.008086*10^{-5}$)	1.04541 (1.09752)	0.3538 (0.4397)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$7.43331*10^{-6}$ ($8.94970*10^{-6}$)	1.43015 (1.37382)	0.9290 (0.9273)
3 Years	$1.19113*10^{-5}$ ($2.24141*10^{-5}$)	1.40258 (1.46189)	0.5975 (0.7150)
5 Years	$2.83098 \cdot 10^{-6}$ ($1.11236*10^{-5}$)	0.69734 (0.58717)	0.2088 (0.1912)
7 Years	$5.77097 \cdot 10^{-6}$ ($3.19880*10^{-5}$)	0.95732 (0.88070)	0.3444 (0.4228)
10 Years	$-1.57533*10^{-5}$ ($-2.93676*10^{-5}$)	0.91349 (0.93195)	0.3612 (0.4532)

* Values in brackets are without 2.5% of max. outliers

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- **Market Information**
- **Three Models for the Pricing of Defaultable Bonds**
  - Schmid and Zagst [2000] (Spread depends on firm-specific factor)
  - **Bakshi, Madan, Zhang [2001]** ***(Spread depends on Treasury rates & firm-specific factor)***
  - Generalized Schmid and Zagst [2004] (Spread depends on macroeconomic & firm-specific factor)
- **Model Comparison**
- **Further Research**

The Model of Bakshi, Madan, Zhang [2001]

Modeling of the Stochastic Processes Under the Martingale Measure Q

- **Dynamics of the yield curve (non-defaultable short rate)**

$$dr(t) = [\omega(t) - a_r \cdot r(t)]\, dt + \sigma_r \sqrt{1-\rho_{r\omega}^2}\, dW_r(t) + \sigma_r \rho_{r\omega} dW_\omega(t) \ , \ t \in [0, T^*],\ a_r > 0,\ \sigma_r > 0,\ |\rho_{r\omega}| < 1$$

- **Dynamics of the unobservable drift driving process**

$$d\omega(t) = \left[\theta_\omega - a_\omega \cdot \omega(t)\right]\, dt + \sigma_\omega dW_\omega(t) \ , \ t \in [0, T^*],\ \theta_\omega \geq 0,\ a_\omega > 0,\ \sigma_\omega > 0$$

- **Dynamics of the unobservable uncertainty index**

$$du(t) = \left[\theta_u - a_u \cdot u(t)\right]\, dt + \sigma_u \frac{\rho_{ru}}{\sqrt{1-\rho_{r\omega}^2}} dW_r(t) + \sigma_u \sqrt{1 - \frac{\rho_{ru}^2}{1-\rho_{r\omega}^2}}\, dW_u(t)$$

$$\theta_u \geq 0,\ a_u > 0,\ \sigma_u > 0,\ \rho_{ru}^2 < 1 - \rho_{r\omega}^2$$

- **The Wiener processes W_r, W_ω, and W_u are uncorrelated**

The Model of Bakshi, Madan, Zhang [2001]

Modeling of the Stochastic Processes Under the Martingale Measure Q

- **The short rate spread is given by**

$$s(t) = \Lambda_0 + (\Lambda_r - 1) \cdot r(t) + \Lambda_u \cdot u(t)\ ,\ t \in [0, T^*],\ \Lambda_0,\ \Lambda_r,\ \Lambda_u \in \mathbb{R},$$

i.e.

$$ds(t) = \left[\theta_s - b_{s\omega} \cdot \omega(t) + b_{su} \cdot u(t) - a_s \cdot s(t)\right] dt + \sigma_s\, dW_s(t)$$

with $\theta_s := \Lambda_0 \cdot a_r + \Lambda_u \cdot \theta_u\ ,\ b_{su} := \Lambda_u \cdot (a_r - a_u)\ ,\ b_{s\omega} := 1 - \Lambda_r\ ,\ a_s = a_r\ ,$

and $\sigma_s\, dW_s(t) := \left((\Lambda_r - 1) \cdot \sigma_r \sqrt{1 - \rho_{r\omega}^2} + \Lambda_u \cdot \sigma_u \frac{\rho_{ru}}{\sqrt{1 - \rho_{r\omega}^2}} \right) dW_r(t) + \Lambda_u \cdot \sigma_u \cdot \sqrt{1 - \frac{\rho_{ru}^2}{1 - \rho_{r\omega}^2}}\, dW_u(t) + (\Lambda_r - 1) \cdot \sigma_r \cdot \rho_{r\omega} dW_\omega(t)$

The Model of Bakshi, Madan, Zhang [2001]
Pricing of Non-Defaultable Bonds

Theorem 5.
The time t value P(r,ω,t,T) of a non-defaultable zero-coupon bond with maturity T≥t is given by

$$P(r,\omega,t,T)=e^{A(t,T)-B(t,T)\cdot r-E(t,T)\cdot\omega}$$

with

$$B(t,T)=\frac{1}{a_r}\cdot\left(1-e^{-a_r\cdot(T-t)}\right),\ E(t,T)=\frac{1}{a_r}\cdot\left(\frac{1-e^{-a_\omega\cdot(T-t)}}{a_\omega}+\frac{e^{-a_\omega\cdot(T-t)}-e^{-a_r\cdot(T-t)}}{a_\omega-a_r}\right),$$

and

$$A(t,T)=\int_t^T\frac{\sigma_r^2}{2}\cdot B^2(\tau,T)+\frac{\sigma_\omega^2}{2}\cdot E^2(\tau,T)+\rho_{r\omega}\cdot\sigma_r\cdot\sigma_\omega\cdot B(\tau,T)\cdot E(\tau,T)-\theta_\omega\cdot E(\tau,T)\,d\tau.$$

The Model of Bakshi, Madan, Zhang [2001]

Pricing of Non-Defaultable Bonds

Theorem 6.

The time t value $P^d(r,u,\omega,t,T)$ of a defaultable zero-coupon bond with maturity $T \geq t$ is given by

$$P^d\left(r,u,\omega,t,T\right)=e^{A^d(t,T)-B^d(t,T)\cdot r-D(t,T)\cdot u-E^d(t,T)\cdot\omega}=P\left(r,\omega,t,T\right)\cdot e^{A^*(t,T)-(\Lambda_r-1)\cdot B(t,T)\cdot r-D(t,T)\cdot u-(\Lambda_r-1)\cdot E(t,T)\cdot\omega}$$

systematic | firm specific | systematic

where

$$B^d\left(t,T\right)=\Lambda_r\cdot B\left(t,T\right),\ D\left(t,T\right)=\tfrac{\Lambda_u}{a_u}\cdot\left(1-e^{-a_u\cdot(T-t)}\right),$$

$$E^d\left(t,T\right)=\Lambda_r\cdot E\left(t,T\right),\ A^*\left(t,T\right)=A^d\left(t,T\right)-A\left(t,T\right),$$

$$\begin{aligned}A^d\left(t,T\right) \quad = \quad & \int_t^T \tfrac{\sigma_r^2}{2}\cdot\Lambda_r^2\cdot B^2\left(\tau,T\right)+\tfrac{\sigma_u^2}{2}\cdot D^2\left(\tau,T\right)+\tfrac{\sigma_\omega^2}{2}\cdot\Lambda_r^2\cdot E^2\left(\tau,T\right)d\tau\\ & +\int_t^T \sigma_r\cdot\sigma_u\cdot\rho_{ru}\cdot\Lambda_r\cdot B\left(t,T\right)\cdot D\left(t,T\right)+\sigma_r\cdot\sigma_\omega\cdot\rho_{r\omega}\cdot\Lambda_r^2\cdot B\left(t,T\right)\cdot E\left(t,T\right)d\tau\\ & -\int_t^T \theta_u\cdot D\left(\tau,T\right)+\theta_\omega\cdot\Lambda_r\cdot E\left(\tau,T\right)-\Lambda_0 d\tau\end{aligned}$$

The Model of Bakshi, Madan, Zhang [2001]
Parameter Estimation

Parameter	Estimation Treasury
a_r ($a_{r,real}$)	0.02235 (0.1149)
ρ_{rw}	-0.31621*
σ_r (%)	1.13146
real-world mean reversion for r (%)	5.62 *(emp. 5.12)*
θ_w(%)	0.11813
a_w ($a_{w,real}$)	0.40316 (0.18272)
σ_w(%)	0.37706
real-world mean reversion for w (%)	0.65

Parameter	Estimation A2	Estimation BBB1
θ_u (%)	0.05292	0.06284
a_u ($a_{u,real}$)	$3.15131 \cdot 10^{-6}$ (0.65739)	$2.57322 \cdot 10^{-6}$ (0.83191)
ρ_{ru}	0.55781	0.58608
σ_u (%)	0.22515	0.26529
real-world mean reversion for u (%)	0.08	0.08
Λ_0	0.01243	0.01671
Λ_r	0.86654	0.83342
real-world mean reversion for short spread (%)	0.57 *(emp. 0.59)*	0.81 *(emp. 0.76)*

* Negative correlation between IR movements and long-term mean (see, e.g. Chen, Scott [93], Dufresne, Solnik [99], Dai, Singleton [00], Bakshi, Madan, Zhang [01]

The Model of Bakshi, Madan, Zhang [2001]

Linear Regression of Model vs. Empirical Treasury Strips*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-1.338 \cdot 10^{-6}$ ($5.619 \cdot 10^{-6}$)	0.85833 (0.8855)	0.82364 (0.8532)
3 Years	$2.261 \cdot 10^{-6}$ ($-3.432 \cdot 10^{-6}$)	1.0269 (1.032)	0.97062 (0.9781)
5 Years	$3.320 \cdot 10^{-6}$ ($-2.198 \cdot 10^{-6}$)	1.04852 (1.0414)	0.97761 (0.9814)
7 Years	$2.994 \cdot 10^{-6}$ ($8.133 \cdot 10^{-6}$)	1.04119 (1.0355)	0.97724 (0.9822)
10 Years	$-2.186 \cdot 10^{-6}$ ($-3.30 \cdot 10^{-6}$)	1.01200 (1.0017)	0.98436 (0.6552)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	-0.000034 (0.000015)	0.74895 (0.8049)	0.67925 (0.8311)
3 Years	$3.997 \cdot 10^{-6}$ (0.000018)	1.04570 (1.044)	0.89052 (0.9193)
5 Years	$6.157 \cdot 10^{-6}$ (0.000014)	1.08862 (1.068)	0.96443 (0.9663)
7 Years	$8.444 \cdot 10^{-6}$ (0.000013)	1.04778 (1.0364)	0.97616 (0.9786)
10 Years	$4.775 \cdot 10^{-6}$ ($-1.926 \cdot 10^{-6}$)	0.99104 (0.9936)	0.98168 (0.9862)

* Values in brackets are without 2.5% of max. outliers

The Model of Bakshi, Madan, Zhang [2001]

Linear Regression of A2-Model vs. Empirical Credit Spreads*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-9.342 \cdot 10^{-6}$ (-0.00001)	1.20747 (1.2179)	0.22376 (0.2882)
3 Years	$-6.360 \cdot 10^{-6}$ ($-5.519 \cdot 10^{-6}$)	1.5829 (1.5158)	0.70011 (0.6913)
5 Years	$-2.7363 \cdot 10^{-6}$ ($8.0168 \cdot 10^{-6}$)	1.22217 (1.1386)	0.43014 (0.4341)
7 Years	$-9.989 \cdot 10^{-6}$ ($-4.710 \cdot 10^{-6}$)	1.21118 (1.1971)	0.41135 (0.4519)
10 Years	$4.7095 \cdot 10^{-6}$ ($4.850 \cdot 10^{-6}$)	1.06099 (1.0874)	0.33634 (0.3953)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$9.7591 \cdot 10^{-6}$ (0.000025)	0.96606 (0.6193)	0.1354 (0.0787)
3 Years	0.0000343 (0.0000243)	2.07913 (1.9753)	0.72088 (0.6493)
5 Years	$2.6660 \cdot 10^{-6}$ ($9.2871 \cdot 10^{-6}$)	1.07221 (0.9101)	0.40694 (0.368)
7 Years	-0.00001262 (0.00001918)	0.73822 (0.6225)	0.23126 (0.2236)
10 Years	-0.00001589 (0.00004427)	0.73198 (0.8275)	0.12251 (0.2083)

* Values in brackets are without 2.5% of max. outliers

The Model of Bakshi, Madan, Zhang [2001]

Linear Regression of BBB1-Model vs. Empirical Credit Spreads*

In sample:

Time to Maturity	a	b	R^2
1 Year	-0.000013 (-0.00002)	1.19189 (1.1335)	0.2350 (0.2769)
3 Years	-0.000011 ($-4.438 \cdot 10^{-6}$)	1.58459 (1.4778)	0.77056 (0.7452)
5 Years	$2.788 \cdot 10^{-6}$ ($7.087 \cdot 10^{-6}$)	0.84195 (0.7392)	0.22994 (0.2428)
7 Years	$6.546 \cdot 10^{-6}$ ($-8.396 \cdot 10^{-6}$)	0.83674 (0.8576)	0.23181 (0.2628)
10 Years	$6.074 \cdot 10^{-6}$ ($6.912 \cdot 10^{-6}$)	0.97741 (0.9851)	0.29005 (0.3515)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$-1.955 \cdot 10^{-6}$ (0.000055)	1.17096 (1.2000)	0.28280 (0.3598)
3 Years	0.000033 (0.000031)	1.89954 (1.7920)	0.80279 (0.7913)
5 Years	$3.657 \cdot 10^{-6}$ (-0.000022)	0.70247 (0.5910)	0.16851 (0.1574)
7 Years	$9.867 \cdot 10^{-6}$ (0.000038)	0.82639 (0.8246)	0.21321 (0.3105)
10 Years	$3.159 \cdot 10^{-6}$ (-0.000013)	0.96050 (0.9598)	0.34311 (0.4069)

* Values in brackets are without 2.5% of max. outliers

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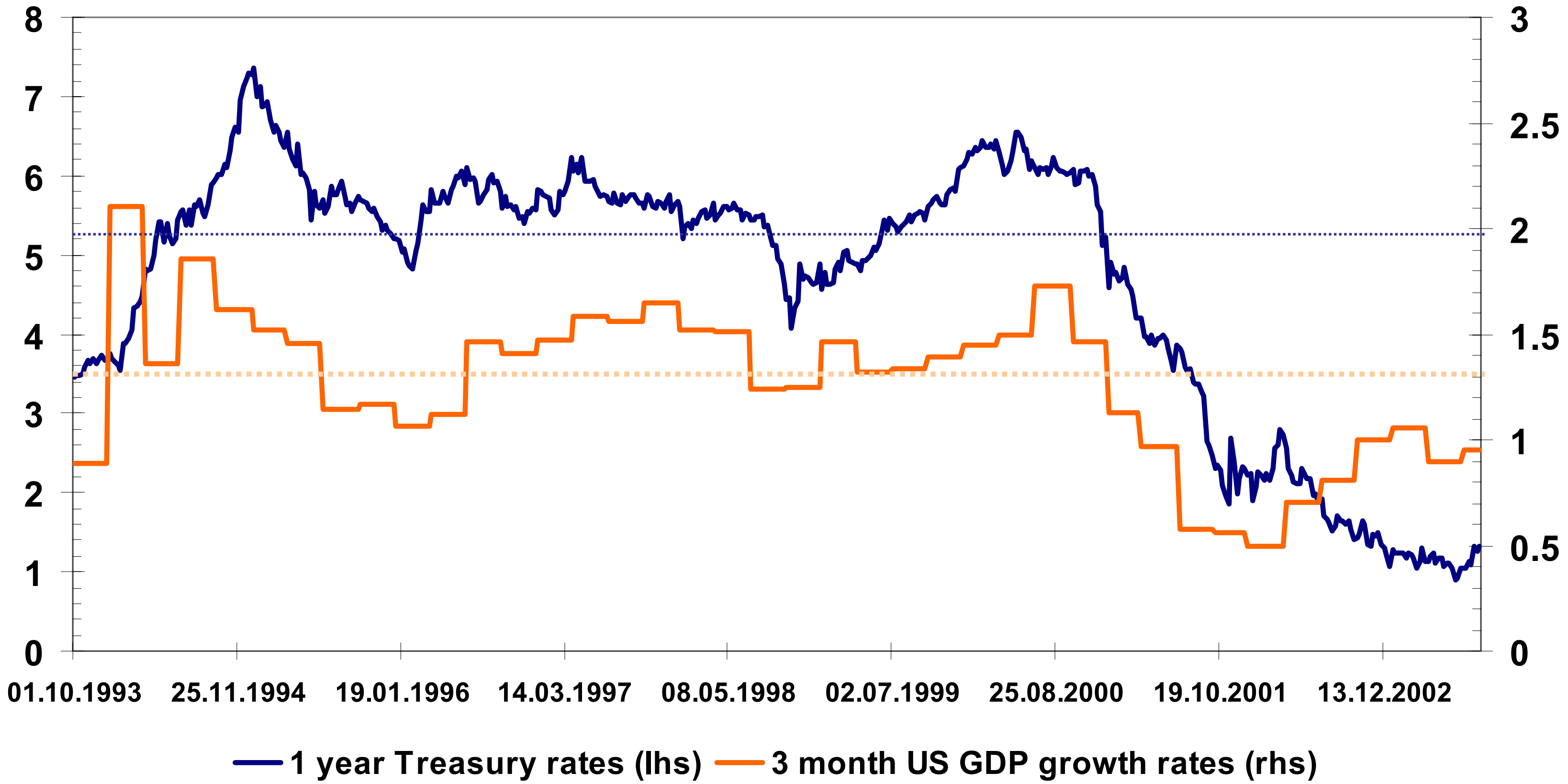
US Treasury Strips and Industrials vs. US Gross Domestic Product

Time Period: 1993 – 2003



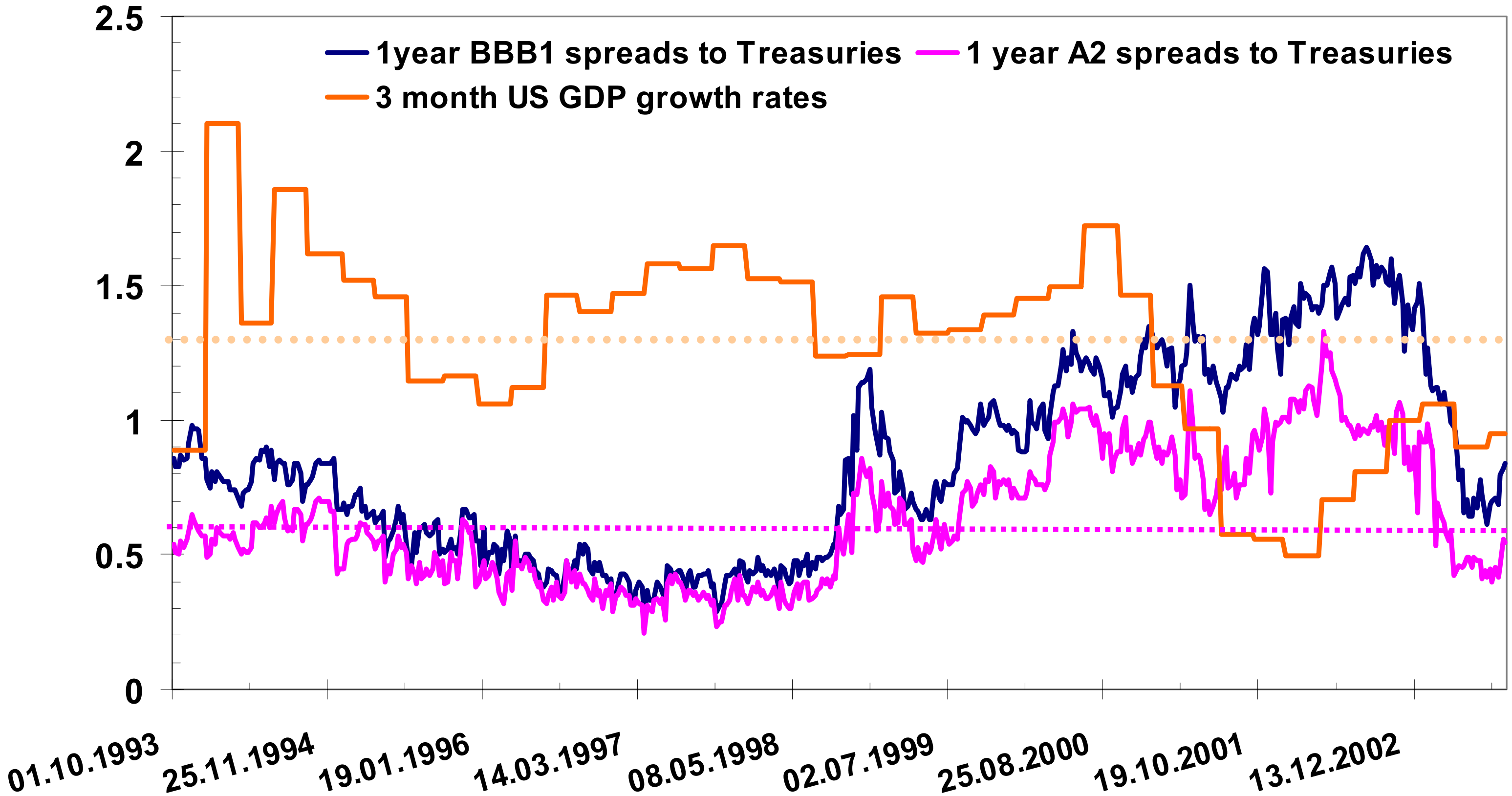
Source: Bloomberg

Linear Regression: $R(t+0.25,t+1.25) = -0.62 + 0.84 \cdot R(t,t+1) + 1.03 \cdot w(t) + \varepsilon$

Correlation (1993-2001): $\rho(R(t,t+1),w(t)) = 0,08$

US Credit Spreads vs. US Gross Domestic Product

Time Period: 1993 – 2003



Linear Regression:

$S^{A2}(t+0.25,t+1.25) = 0.25 + 0.84 \cdot S^{A2}(t,t+1) - 0.10 \cdot w(t) + \varepsilon$

$S^{BBB1}(t+0.25,t+1.25) = 0.24 + 0.77 \cdot S^{BBB1}(t,t+1) - 0.08 \cdot w(t) + \varepsilon$

Source: Bloomberg

Correlation (1993-2001): $\rho(S^{A2}(t,t+1),w(t)) = -0{,}11$, $\rho(S^{BBB1}(t,t+1),w(t)) = -0{,}15$

Correlation (1993-2002): $\rho(w(t),\text{default probability BBB}) = -0{,}49$

The Generalized Model of Schmid and Zagst [2004]

Modeling of the Stochastic Processes Under the Martingale Measure Q

- **Dynamics of the yield curve (non-defaultable short rate)**

$$dr(t) = [\theta_r(t) + b_{r\omega} \cdot \omega(t) - a_r \cdot r(t)]dt + \sigma_r dW_r(t),\ t \in [0, T^*],\ a_r > 0, \sigma_r > 0$$

- **Dynamics of the economy index**

$$d\omega(t) = [\theta_\omega - a_\omega \cdot \omega(t)]dt + \sigma_\omega dW_\omega(t),\ t \in [0, T^*],\ \theta_\omega \geq 0, a_\omega > 0, \sigma_\omega > 0$$

- **Dynamics of the uncertainty index**

$$du(t) = [\theta_u - a_u \cdot u(t)]dt + \sigma_u dW_u(t), t \in [0, T^*],\ \theta_u \geq 0, a_u > 0, \sigma_u > 0$$

- **Dynamics of the yield spread (short-rate credit spread)**

$$ds(t) = [\theta_s + b_{su} \cdot u(t) - b_{s\omega} \cdot \omega(t) - a_s \cdot s(t)]dt + \sigma_s dW_s(t),\ t \in [0, T^*],\quad \begin{matrix} \theta_s \geq 0, b_{su} > 0, b_{s\omega} > 0, \\ a_s > 0, \sigma_s > 0 \end{matrix}$$

- **The Wiener processes W_r , W_ω , W_u , and W_s are uncorrelated**

Theorem 7.
The time t value P(r,ω,t,T) of a non-defaultable zero-coupon bond with maturity T≥t is given by

$$P(r,\omega,t,T)=e^{A(t,T)-B(t,T)\cdot r-E(t,T)\cdot\omega}$$

with

$$B(t,T)=\frac{1}{a_r}\cdot\left(1-e^{-a_r\cdot(T-t)}\right),\ E(t,T)=\frac{b_{r\omega}}{a_r}\cdot\left(\frac{1-e^{-a_\omega\cdot(T-t)}}{a_\omega}+\frac{e^{-a_\omega\cdot(T-t)}-e^{-a_r\cdot(T-t)}}{a_\omega-a_r}\right),$$

and

$$A(t,T)=\int_t^T\frac{\sigma_r^2}{2}\cdot B^2(\tau,T)+\frac{\sigma_\omega^2}{2}\cdot E^2(\tau,T)-\theta_r(\tau)\cdot B(\tau,T)-\theta_\omega\cdot E(\tau,T)\,d\tau.$$

The Generalized Model of Schmid and Zagst [2004]

Pricing of Non-Defaultable Bonds

Theorem 8.

The time t value $P^d(r,s,u,\omega,t,T)$ of a defaultable zero-coupon bond with maturity $T \geq t$ is given by

$$
\begin{aligned}
P^d(r,s,u,\omega,t,T) &= e^{A^d(t,T)-B(t,T)\cdot r-C(t,T)\cdot s-D(t,T)\cdot u-E^d(t,T)\cdot\omega} \\
&= P(r,\omega,t,T)\cdot e^{A^*(t,T)-C(t,T)\cdot s-D(t,T)\cdot u+E^*(t,T)\cdot\omega}
\end{aligned}
$$

systematic — firm specific — systematic

where

$$C(t,T)=\frac{1}{a_s}\cdot\left(1-e^{-a_s\cdot(T-t)}\right),\ D(t,T)=\frac{b_{su}}{a_s}\cdot\left(\frac{1-e^{-a_u\cdot(T-t)}}{a_u}+\frac{e^{-a_u\cdot(T-t)}-e^{-a_s\cdot(T-t)}}{a_u-a_s}\right),$$

$$E^d(t,T)=E(t,T)-E^*(t,T)\ \text{with}\ E^*(t,T)=\frac{b_{s\omega}}{a_s}\cdot\left(\frac{1-e^{-a_\omega\cdot(T-t)}}{a_\omega}+\frac{e^{-a_\omega\cdot(T-t)}-e^{-a_s\cdot(T-t)}}{a_\omega-a_s}\right),$$

$$A^d(t,T)=A(t,T)+A^*(t,T)\ \text{with}$$

$$
\begin{aligned}
A^*(t,T) &= \int_t^T \frac{\sigma_s^2}{2}\cdot C^2(\tau,T)+\frac{\sigma_u^2}{2}\cdot D^2(\tau,T)+\frac{\sigma_\omega^2}{2}\cdot {E^*}^2(\tau,T)-\sigma_\omega^2\cdot E(\tau,T)\cdot E^*(\tau,T)\,d\tau \\
&\quad -\int_t^T \theta_s\cdot C(\tau,T)+\theta_u\cdot D(\tau,T)-\theta_\omega\cdot E^*(\tau,T)\,d\tau
\end{aligned}
$$

The Generalized Model of Schmid and Zagst [2004]

Parameter Estimation

Parameter	Estimation Treasury
a_r ($a_{r,real}$)	0.04452 (0.07126)
b_{rw}	0.20313
σ_r (%)	1.29036
real-world mean reversion level for r (%)	5.21 *(emp. 5.12)*
θ_w (%)	1.85687
a_w ($a_{w,real}$)	0.54521 (1.41388)
σ_w(%)	0.79100
real-world mean reversion level for w (%)	1.31 *(emp. 1.37)*

Parameter	Estimation A2	Estimation BBB1
a_s ($a_{s,real}$)	1.05334 (2.49603)	1.01934 (2.16216)
θ_s (%)	1.33644	1.39585
σ_s (%)	0.65233	0.56953
b_{sw}	0.00395	0.00361
real-world mean reversion level for s (%)	0.60 *(emp. 0.59)*	0.74 *(emp. 0.76)*
θ_u (%)	0.03037	0.03444
a_u ($a_{u,real}$)	$3.31995 \cdot 10^{-6}$ (0.17408)	$1.01701 \cdot 10^{-6}$ (0.16993)
σ_u (%)	0.20018	0.19795
real-world mean reversion level for u (%)	0.17	0.20

The Generalized Model of Schmid and Zagst [2004]

Linear Regression of Model vs. Empirical Treasury Strips*

In sample:

Time to Maturity	a	b	R^2
1 Year	-0.000021 (-0.000011)	0.74124 (0.7518)	0.70954 (0.76025)
3 Years	$-9.747 \cdot 10^{-6}$ ($-1.187 \cdot 10^{-6}$)	0.93959 (0.9577)	0.95092 (0.9615)
5 Years	$-6.256 \cdot 10^{-8}$ ($-2.001 \cdot 10^{-7}$)	1.00896 (1.0078)	0.99972 (0.9999)
7 Years	$6.449 \cdot 10^{-6}$ ($5.622 \cdot 10^{-6}$)	1.00989 (1.009)	0.94460 (0.9556)
10 Years	$7.863 \cdot 10^{-6}$ ($-5.766 \cdot 10^{-6}$)	1.01219 (1.0085)	0.91327 (0.9342)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	-0.000132 (-0.000089)	0.65090 (0.7109)	0.62117 (0.7026)
3 Years	-0.000051 (-0.000037)	0.94388 (0.8997)	0.91663 (0.9256)
5 Years	$8.572 \cdot 10^{-7}$ ($1.471 \cdot 10^{-6}$)	1.01041 (1.0087)	0.99987 (0.9999)
7 Years	0.000033 (0.000037)	0.98171 (1.0077)	0.95381 (0.9666)
10 Years	0.000053 (0.000037)	0.95137 (1.0016)	0.89738 (0.9349)

* Values in brackets are without 2.5% of max. outliers

The Generalized Model of Schmid and Zagst [2003]

Linear Regression of A2-Model vs. Empirical Credit Spreads*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-2.088 \cdot 10^{-6}$ ($-6.831 \cdot 10^{-6}$)	1.31997 (1.2612)	0.8966 (0.8987)
3 Years	$5.354 \cdot 10^{-7}$ ($1.724 \cdot 10^{-6}$)	1.0917 (1.0683)	0.4526 (0.4828)
5 Years	$-6.284 \cdot 10^{-6}$ ($4.703 \cdot 10^{-8}$)	1.3654 (1.3529)	0.5564 (0.5806)
7 Years	$-6.459 \cdot 10^{-6}$ ($5.04 \cdot 10^{-8}$)	1.4064 (1.3602)	0.5307 (0.5494)
10 Years	$1.22 \cdot 10^{-6}$ (0.000011)	1.1073 (1.1520)	0.3391 (0.4044)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$7.218 \cdot 10^{-6}$ ($8.434 \cdot 10^{-7}$)	1.3765 (1.3162)	0.9111 (0.9006)
3 Years	$8.249 \cdot 10^{-6}$ (0.000021)	1.3617 (1.3201)	0.3554 (0.3781)
5 Years	0.000016 (0.000019)	1.4094 (1.2297)	0.5765 (0.5358)
7 Years	$1.092 \cdot 10^{-6}$ (0.000012)	1.0820 (1.0088)	0.3764 (0.4152)
10 Years	$-9.987 \cdot 10^{-6}$ (0.000044)	0.9113 (0.9345)	0.1424 (0.1975)

* Values in brackets are without 2.5% of max. outliers

The Generalized Model of Schmid and Zagst [2004]

Linear Regression of BBB1-Model vs. Empirical Credit Spreads*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-4.803 \cdot 10^{-6}$ $(-6.061 \cdot 10^{-6})$	1.48118 (1.4005)	0.87491 (0.8712)
3 Years	$-3.133 \cdot 10^{-6}$ $(3.648 \cdot 10^{-6})$	1.16911 (1.1591)	0.46308 (0.5129)
5 Years	$-9.269 \cdot 10^{-7}$ $(-2.041 \cdot 10^{-6})$	0.88573 (0.7622)	0.2247 (0.2187)
7 Years	$-1.473 \cdot 10^{-6}$ $(-5.707 \cdot 10^{-6})$	1.00664 (0.9573)	0.27675 (0.2854)
10 Years	$-4.782 \cdot 10^{-6}$ $(6.201 \cdot 10^{-6})$	1.18693 (1.3227)	0.34191 (0.4549)

Out of sample:

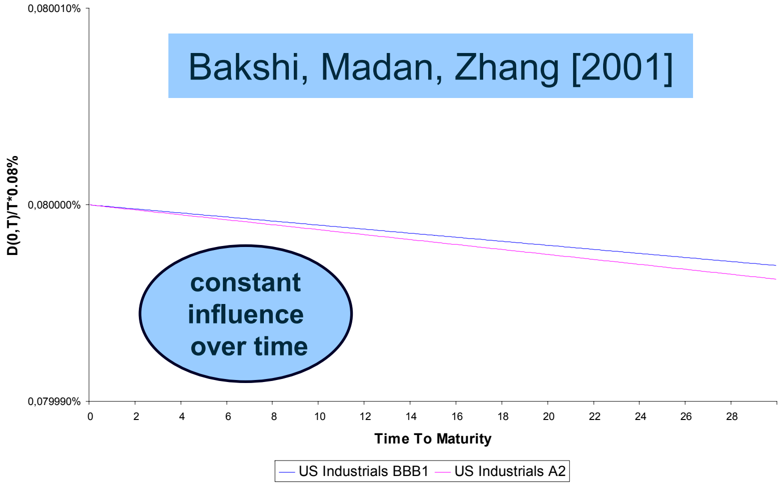
Time to Maturity	a	b	R^2
1 Year	0.000017 (0.000018)	1.52451 (1.457)	0.90531 (0.9057)
3 Years	0.000022 (0.000028)	1.52953 (1.4841)	0.47917 (0.5246)
5 Years	$5.474 \cdot 10^{-6}$ (-0.000022)	0.80833 (0.6675)	0.15912 (0.1444)
7 Years	0.000021 (0.000046)	1.21613 (1.1625)	0.2971 (0.3998)
10 Years	$1.292 \cdot 10^{-6}$ $(6.091 \cdot 10^{-6})$	1.22823 (1.0871)	0.33841 (0.3560)

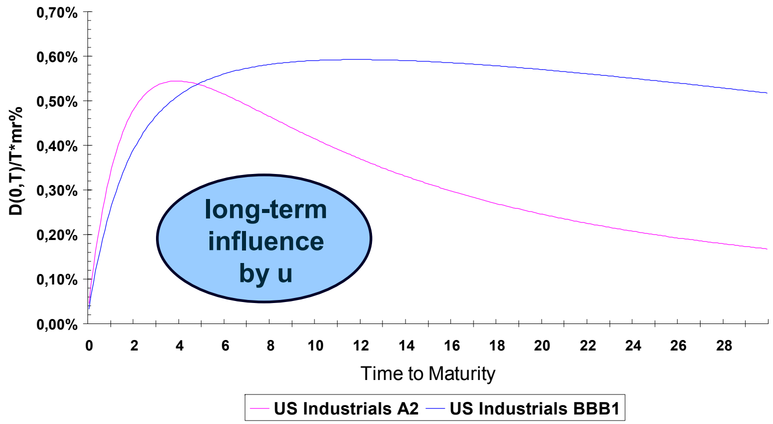
* Values in brackets are without 2.5% of max. outliers

A New Model for the Pricing of Defaultable Bonds
Overview

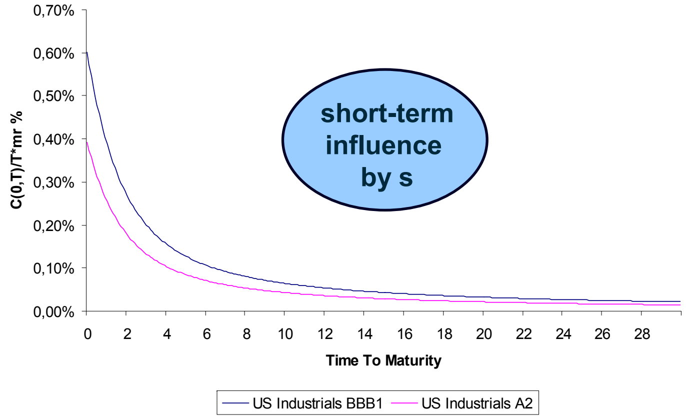
- **Market Information**
- **Three Models for the Pricing of Defaultable Bonds**
- **Model Comparison**
  - Structural Information
  - Statistical Tests
  - Default Probabilities
- **Further Research**

Model Comparison
Sensitivity of Credit Spreads with Respect to Firm-specific Factors



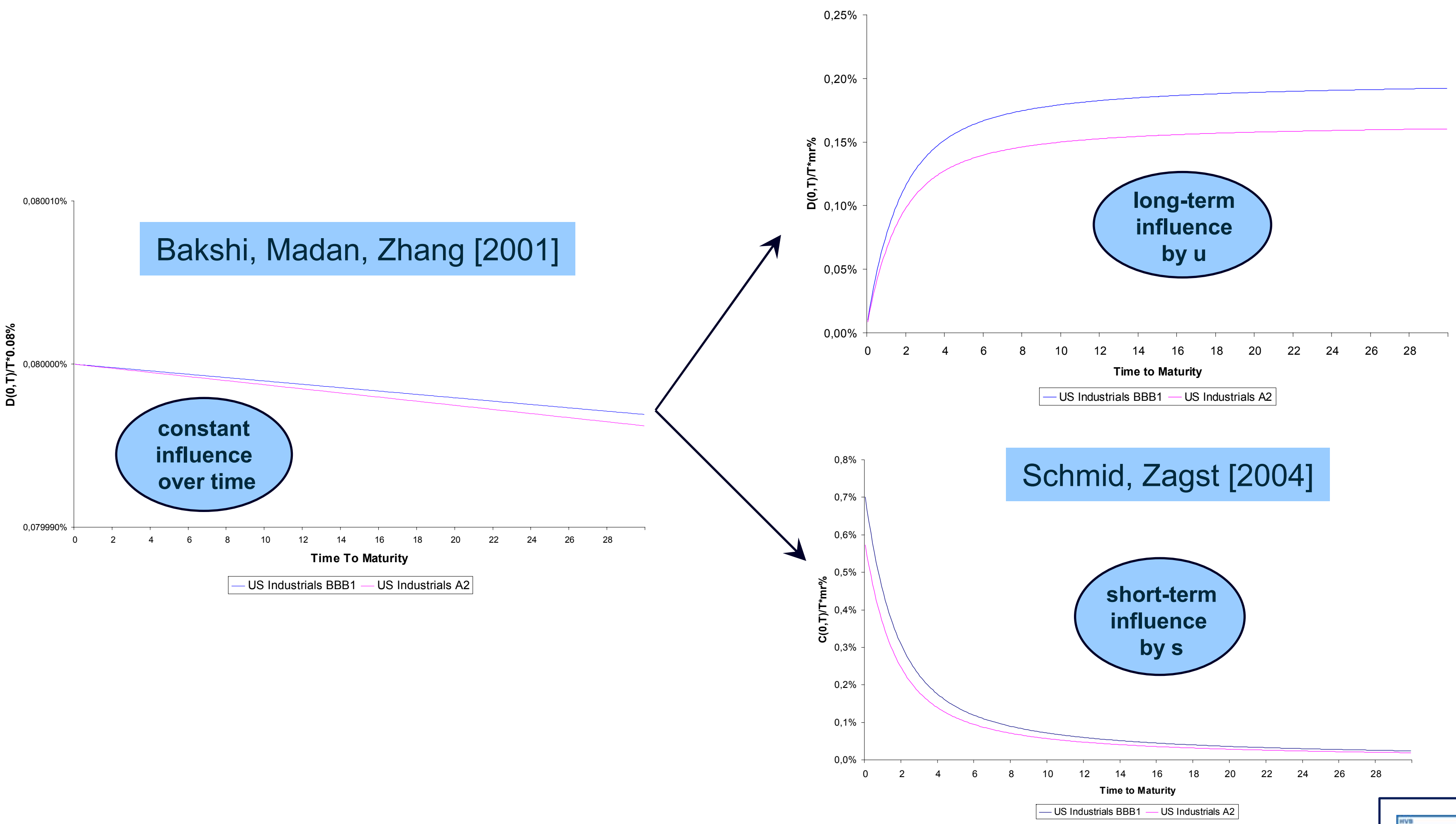


Schmid, Zagst [2000]



Model Comparison

Sensitivity of Credit Spreads with Respect to Firm-specific Factors



Model Comparison
Average Absolute Deviation of Treasury Strips, Credit Spreads and R^2

Average absolute Deviation of yields / spreads (%)	Schmid, Zagst [2000]	Bakshi, Madan, Zhang [2001]	Schmid, Zagst [2004]
Treasury Strips	0.18687 (0.84627)	0.11415 (0.19366)	0.17404 (0.91265)
Industrials A2	0.09970 (0.10133)	0.08715 (0.16887)	0.06212 (0.08976)
Industrials BBB1	0.08742 (0.14296)	0.10812 (0.23154)	0.08229 (0.16423)

Average R^2	Schmid, Zagst [2000]	Bakshi, Madan, Zhang [2001]	Schmid, Zagst [2004]
Treasury Strips	0.74268 (0.73733)	0.87807 (0.8242)	0.81724 (0.76946)
Industrials A2	0.29696 (0.29966)	0.34563 (0.26440)	0.46382 (0.42243)
Industrials BBB1	0.37632 (0.39361)	0.30582 (0.32646)	0.42623 (0.41639)

* Values without brackets are in sample, values in brackets are out of sample

Model Comparison

Does the Introduction of Positive Spreads or GDP Pay?

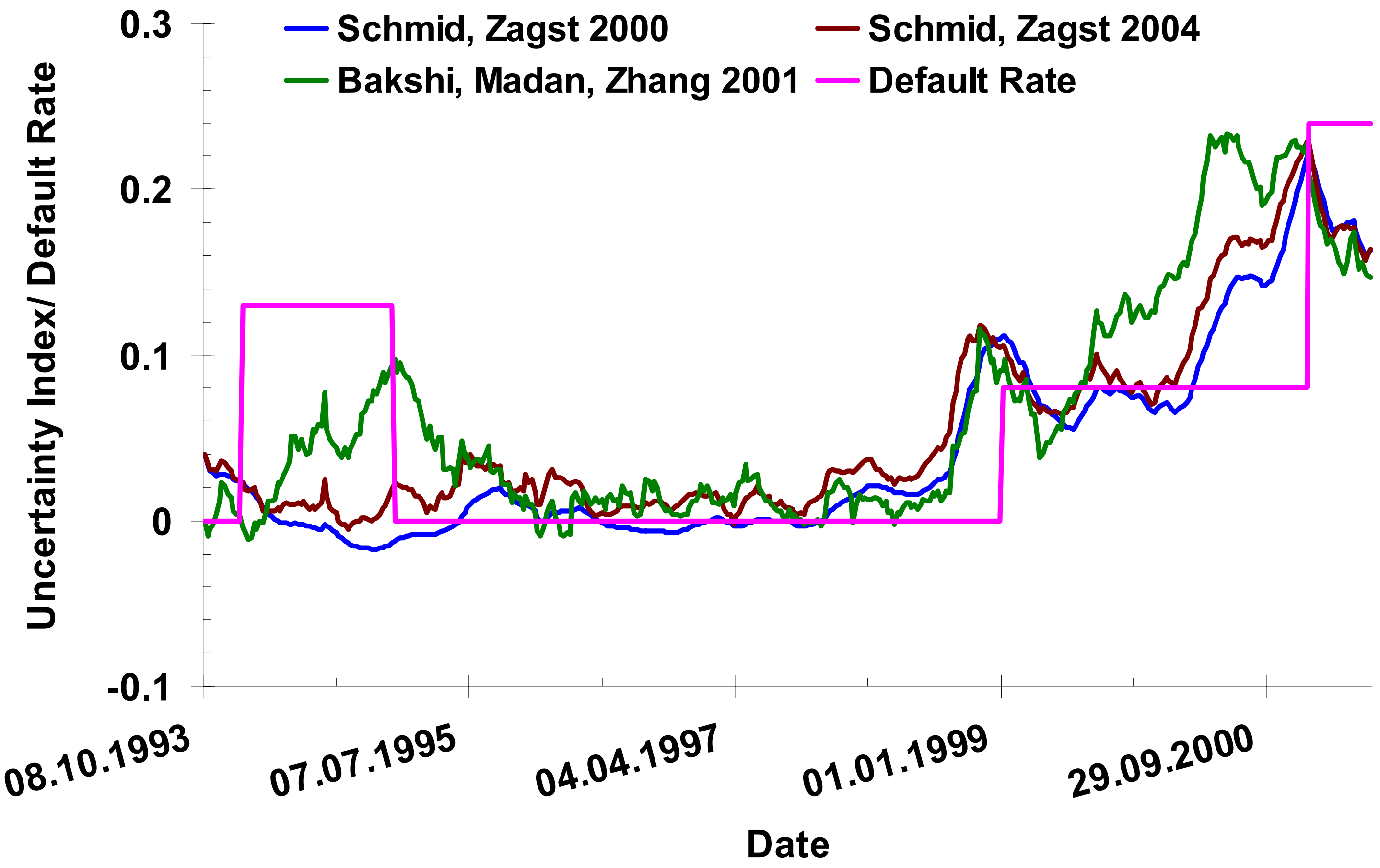
Average absolute deviation of yields / spreads (%)	***Positive spreads,*** no macro variable (SZ 2000)	Normal spreads, no macro variable ($b_{rw}=b_{sw}=0$)	Normal spreads, ***macro variable*** (SZ 2004)
Treasury Strips	0.18687 (0.84627)	0.18687 (0.84627)	0.17404 (0.91265)
Industrials A2	0.09970 (0.10133)	0.08206 (0.16375)	0.06212 (0.08976)
Industrials BBB1	0.08742 (0.14296)	0.07654 (0.08798)	0.08229 (0.16423)

Average R^2	***Positive spreads,*** no macro variable (SZ 2000)	Normal spreads, no macro variable ($b_{rw}=b_{sw}=0$)	Normal spreads, ***macro variable*** (SZ 2004)
Treasury Strips	0.74268 (0.73733)	0.74268 (0.73733)	0.81724 (0.76946)
Industrials A2	0.29696 (0.29966)	0.38941 (0.38514)	0.46382 (0.42243)
Industrials BBB1	0.37632 (0.39361)	0.35811 (0.32023)	0.42623 (0.41639)

* Values without brackets are in sample, values in brackets are out of sample

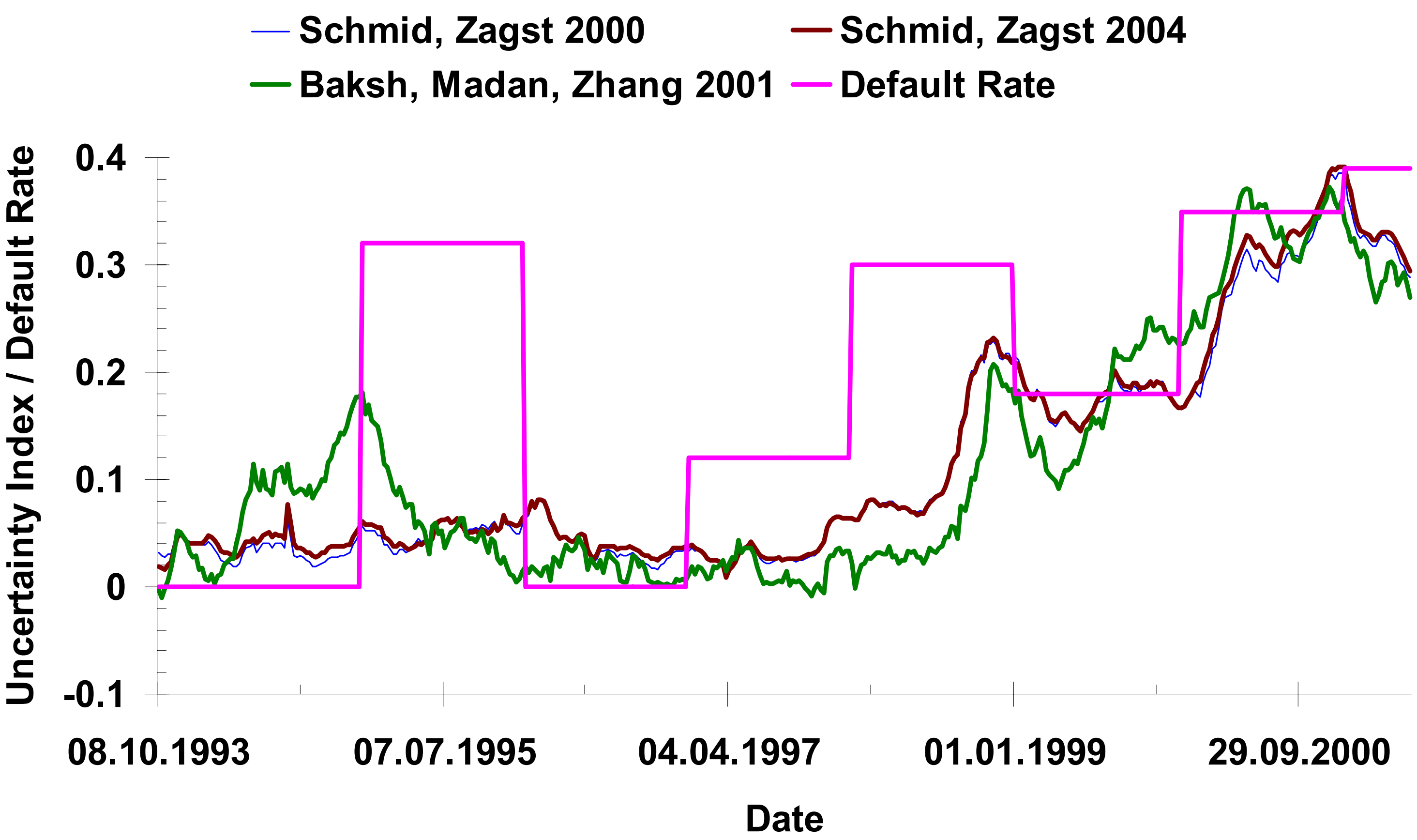
Model Comparison

Uncertainty Index vs. Default Probability for Rating A2



Model Comparison

Uncertainty Index vs. Default Probability for Rating BBB1



A New Model for the Pricing of Defaultable Bonds
Overview

- **Market Information**
- **Three Models for the Pricing of Defaultable Bonds**
- **Model Comparison**
- **Further Research**
  - Inclusion of Other Models (Duffee [1996])
  - Integration of Other Macroeconomic Factors (Inflation)
  - Pricing of Credit Derivatives (Credit Default Swaps)
  - Forecast and Simulation of Transition (Default) Probabilities
  - Integrated Market and Credit Portfolio Management

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