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## Dynamic Portfolio Credit Risk : Credit Derivatives, Credit Spreads and Infections in Credit Portfolios

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# Agenda

**Dynamic Credit Risk Modelling**

**Observations of Dynamic Portfolio effects**

**Credit Spreads, Infections and Frailties**

**Evolutionary Portfolio Credit Management**

**Dynamic Credit Risk Modelling**

# Realistic Risk Modelling

- Credit Risk Management needs a measure for :
  - Economic and regulatory capital calculations (Basel II A-IRB)
  - Loss reserving
  - Variability of losses
- The need to model key Risk measures such as :
  - Forward looking cycle average defaults
  - Dependence among defaults
  - Variation of the default probability over the credit cycle
  - Practitioners have to choose a functional form and estimate the parameters of the chosen model
  - Data limitations . Historical Data . Standard error around parameter estimates

**THE CHOICE OF A REALISTIC MODEL IS A POSTULATION**

# Loss distribution

- Vasicek ( 1987,1997 ) . Inverse Gaussian distribution .
- Basically the values of the assets of an obligor are driven by one common factor Y and an idiosyncratic standard normal noise  $\varepsilon_i$  where :

$$V_i(T) = \sqrt{\rho}Y + \sqrt{1-\rho}\varepsilon_i$$

- The probability that the fraction L of defaults in the portfolio is less than a given level q is given by :

$$P[L \leq q] = \Phi\left(\frac{1}{\sqrt{\rho}}(\sqrt{1-\rho}\Phi^{-1}(q) - \Phi^{-1}(p))\right)$$

- Where :
- p is the default probability of any obligor in the portfolio
- $\rho$  is the asset value correlation between any two obligors .
- $\Phi$  is the cumulative standard normal distribution

- Normally the loss distribution will need very long numerical simulations
- Credit Metrics model can be seen as a more general model of which Vasicek is a limiting case ( With  $X_i := \Phi(V_i)$  and  $p := \Phi(K)$  )
- An interesting relationship is the conditional Probability of Default given the common factor Y( Also from Vasicek ) :

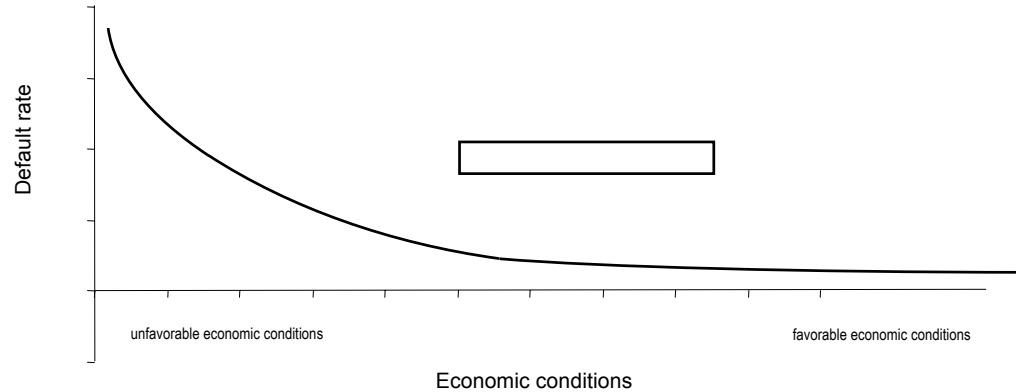
$$p|_Y = \Phi\left[\Phi^{-1}(\bar{p}) - \frac{\sqrt{\rho}Y}{\sqrt{1-\rho}}\right] = \Phi\left[c - \frac{\sqrt{\rho}Y}{\sqrt{1-\rho}}\right]$$

Where m is standard normal distributed

$\Phi[c] = \bar{p}$  with  $\bar{p}$  as unconditional default probability

# Loss distribution

## Default rates and single factor approach Y

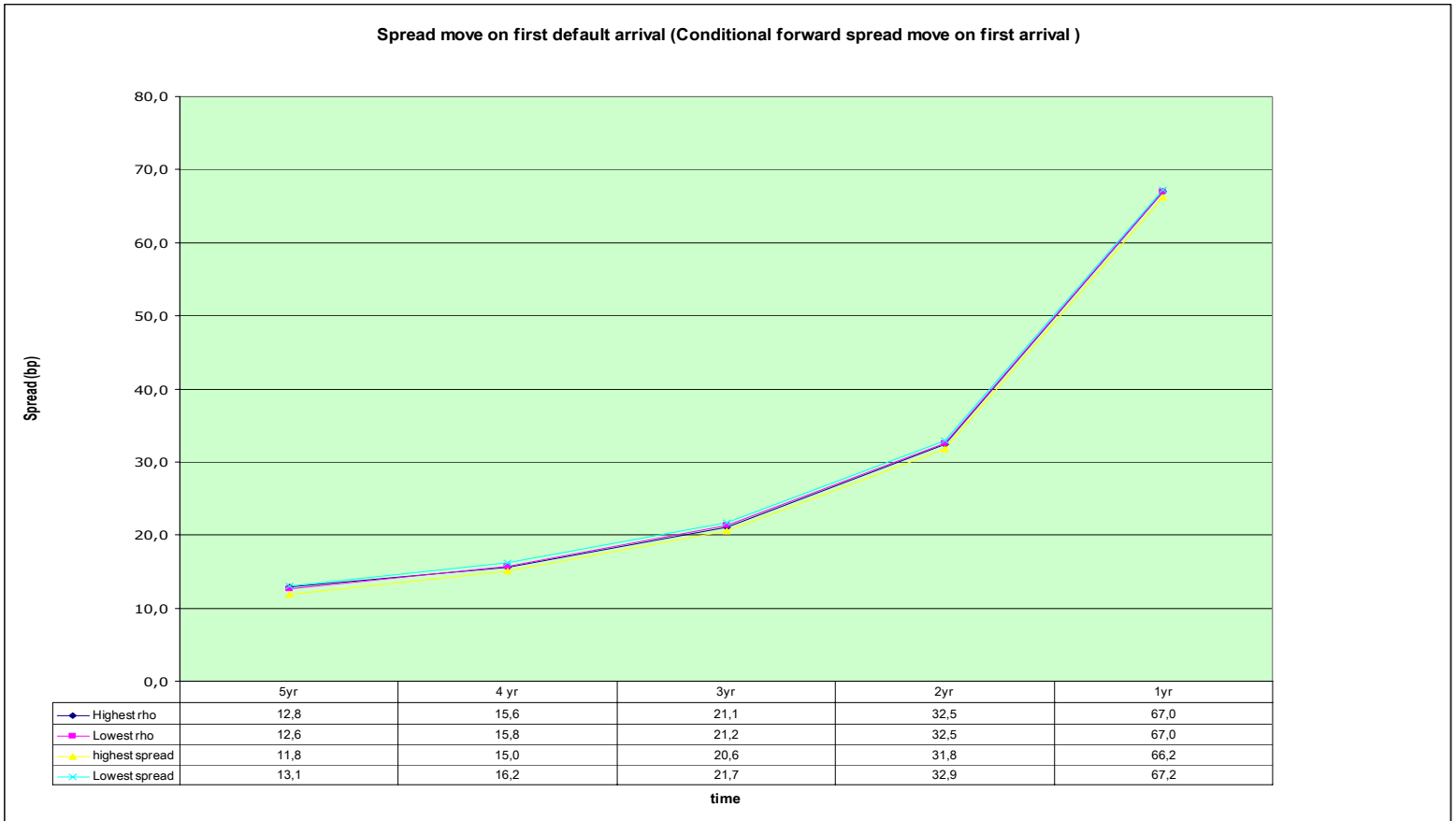


- Correlation is in the curvature of the transformation function ( The conditional probability of default given  $Y$  ) .
- Difficulties replicating the loss distribution for CDO's given that only  $\rho$  will be the variable to fit the main parts of the distribution : mezzanine, equity and senior tranches . Not enough degrees of freedom to match the loss distributions observed in the market
- Furthermore the shape of the distribution changes significantly when different time horizons ( and thus default probabilities ) are considered .
- We will not use then the multivariate normally distributed random variables to trigger default instead we will use Archimedean copula functions .

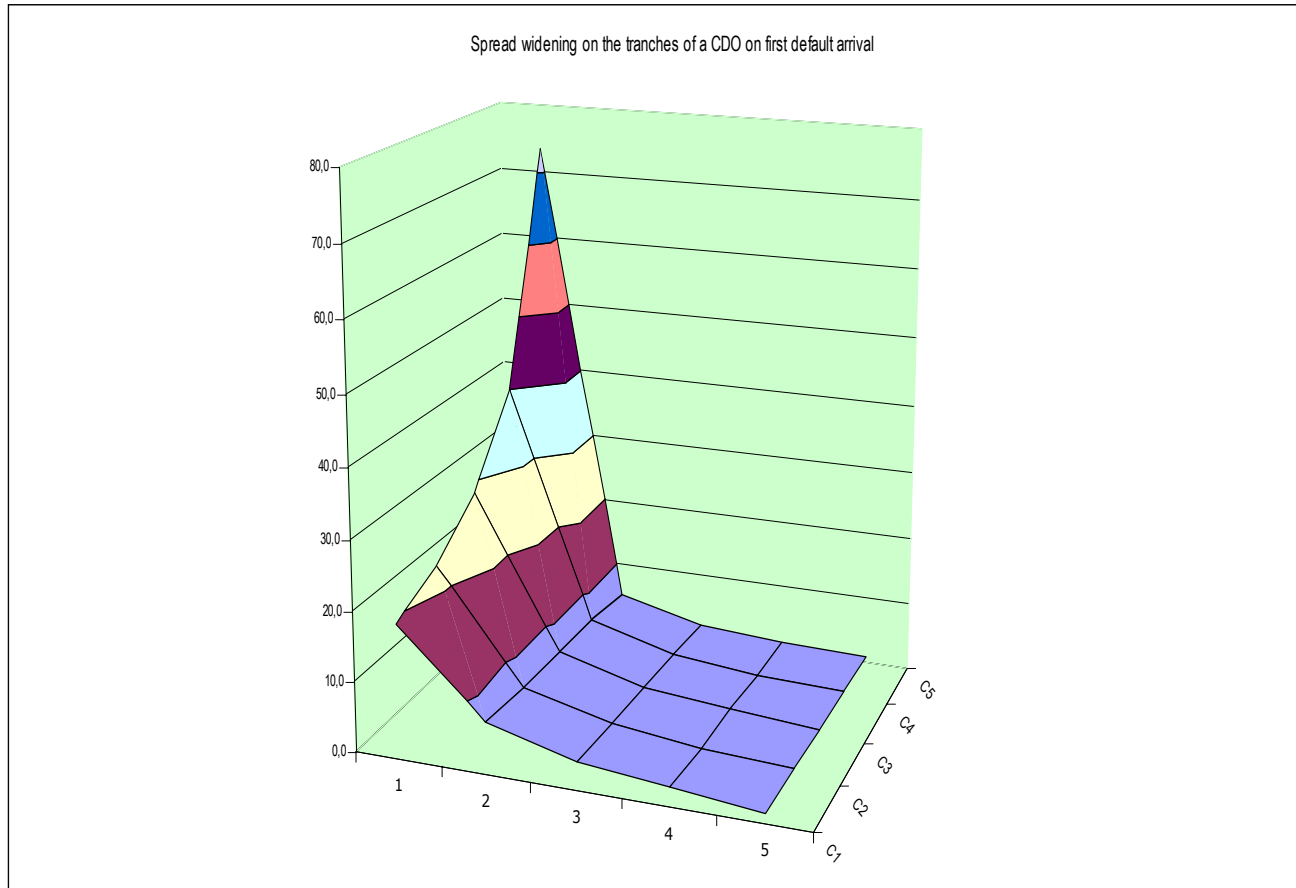
**TRANSFORMATION FUNCTION CAN NOT PRICE GENERAL CDO DYNAMICS**

**Observations of Dynamic Portfolio effects**

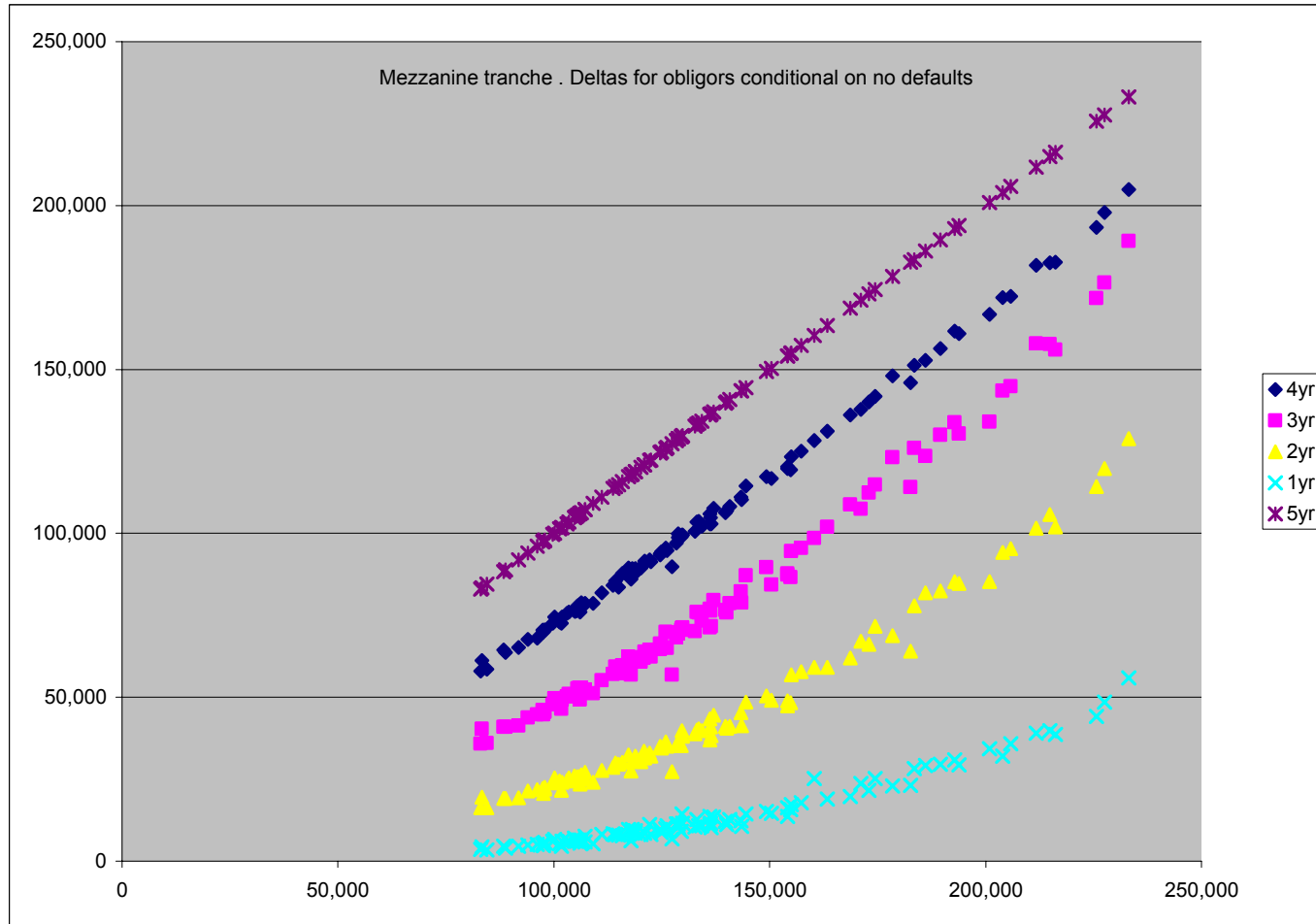
# Dynamic Portfolio effects



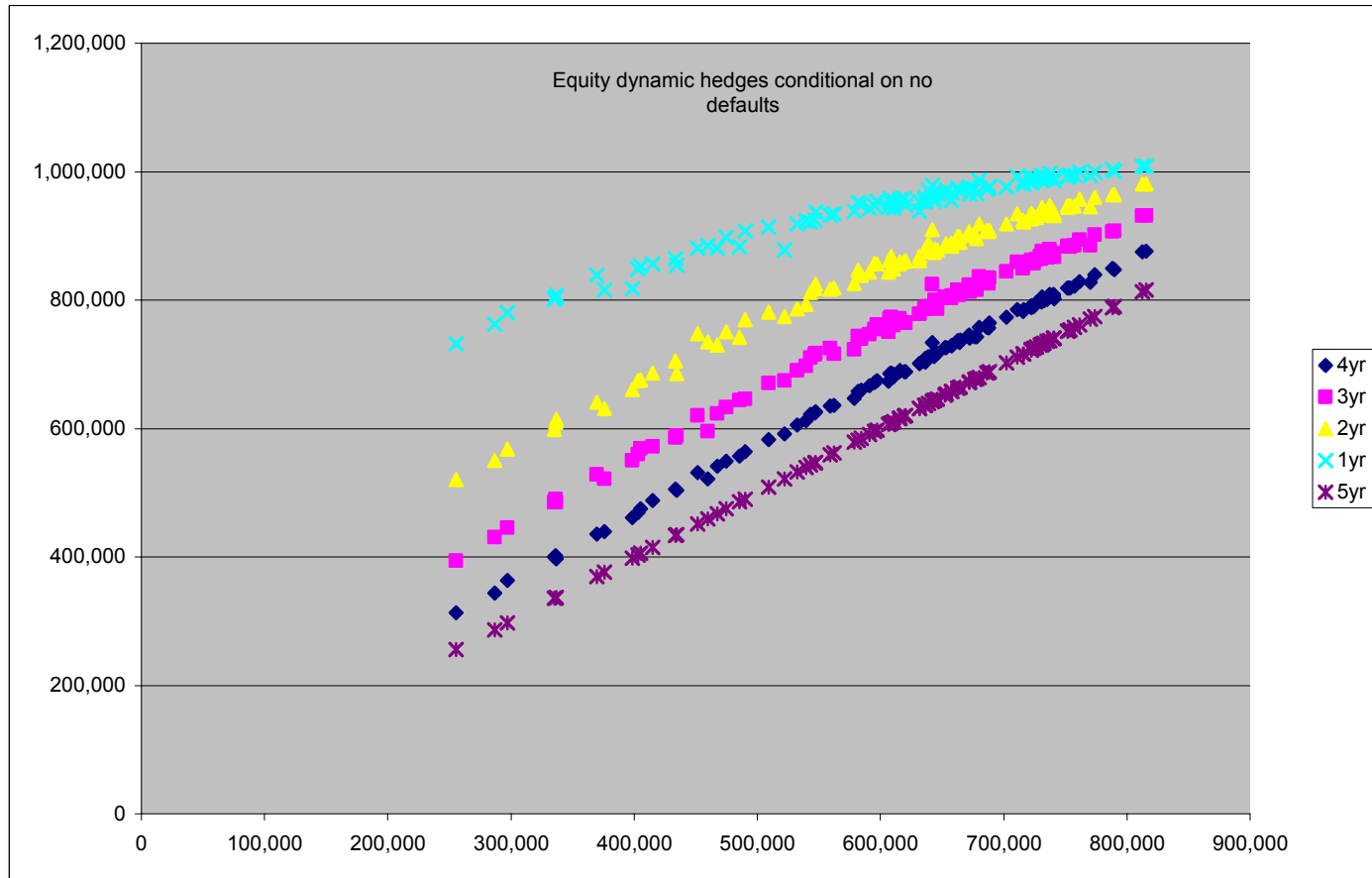
# Dynamic Portfolio effects



# Dynamic Portfolio effects



# Dynamic Portfolio effects



**Credit Spreads, Infections and Frailties**

# The Frailty

- Two obligors  $i=1,2$ . Both defaults will be triggered by a Poisson process of intensity  $h_i$ . Default times  $\tau_i, i=1,2$ .
- The survival probabilities of the obligors are  $P_i(t,T)=e^{-(T-t)h_i}$  if both Poisson are independent then the joint survival function is :

$$P(t, T_1, T_2) = P(\tau_1 > T_1, \tau_2 > T_2) = e^{-(T_1-t)h_1 - (T_2-t)h_2} \quad (1)$$

Let's assume that  $\mathbf{h}_1 = \mathbf{Y} \lambda_1$  and  $\mathbf{h}_2 = \mathbf{Y} \lambda_2$ , (2), where  $Y$  is a continuous random variable with density  $f_Y(y)$ . Conditional to  $Y$  the default processes  $N_1(t)$  and  $N_2(t)$  are still independent processes with intensities  $h_1$  and  $h_2$ .  $Y$  is called a **frailty** in biostatistics (Hougaard 2000). Normally the frailty of patients in a group and usually unobserved common influence. With initial value of the expectation of  $Y$ ,  $E[Y]=1$ .

We can interpret the frailty as the **uncertainty** around  $\lambda_1$ .

We could have chosen a more general case for the model in the econometric literature ie

$$h_i = e^{\left[ \sum_{n=1}^N x_{in} \beta_n + \varepsilon_i \right]} = e^{\left[ \sum_{n=1}^N x_{in} \beta_n \right]} e^{\left[ \sum_{n=1}^N \varepsilon_i \right]} = Y_i e^{\left[ \sum_{n=1}^N x_{in} \beta_n \right]} \quad (3)$$

$x_{in}$  With is  $n^{\text{th}}$  covariate of the  $i^{\text{th}}$  obligor

$\beta_n$  Are coefficients of the influence of  $x_{in}$

- **Initial Survival Probabilities**
- Using the same frailty for both we can say that they are independent if conditioned on the frailty  $Y$ . Therefore the model shows **conditional independence**.

Conditional on  $Y=y$  we have the initial joint survival function  $P[\tau_1 > T_1 \wedge \tau_2 > T_2 | Y = y] = e^{[-y(\lambda_1 T_1 + \lambda_2 T_2)]}$  (5) with  $\tau_i$  the time to default of obligor  $i$

# The Frailty

The joint conditional density function of the default times is

$$f(T_1, T_2 | y) = y^2 \lambda_1 \lambda_2 e^{-y(\lambda_1 T_1 + \lambda_2 T_2)} \quad (6)$$

- Taking the expectation of the conditional joint probability we obtain the joint survival probability and the individual survival probabilities :

$$E[P[\tau_1 > T_1 \wedge \tau_2 > T_2]] = E[e^{-y(\lambda_1 T_1 + \lambda_2 T_2)}] \quad P[\tau_i > T_i] = E[e^{-Y \lambda_i T_i}] \quad (7)$$

- And the initial default hazard rate

$$h_i(0) = -\frac{\partial}{\partial T} E[e^{-Y \lambda_i T}] |_{T=0} = \lambda_i E[Y] \quad (8)$$

- If the Laplace transform is known in close form :

$$L_Y(s) := E[e^{-sY}] , \quad (9)$$

- Determines the distribution of Y and we can easily determine the expectations with respect to Y easily .
- Comparing equations 7 and 9 with 1 the introduction of the frailty is a replacement of the exponential function with the Laplace transform function . The exponential function is itself a Laplace function . If we choose Y to be  $\Gamma(\alpha, \beta)$  then the Laplace transform is  $(1 + \beta s)^{-\alpha}$  . (10)

# The Frailty

- **Survival Probabilities at later times** .
- As time goes by and no defaults happen we gather information about the likely values of the frailty  $Y$  . We assume that we only have new information about observations of defaults or survival of the obligors . The result is that we should revise the distribution of  $Y$  if no defaults happen to making the values of the frailty lower . The contrary should happen too if we observe a default then there should be a high frailty we have under-estimated the value of  $Y$  .
- **Updating the frailty**
- Equation (6) gives us the joint conditional density of the default times and the frailty and (5) the conditional distribution . From this we will derive the conditional density of  $y$  , conditional on the observations that we have made on  $\tau_1$  and  $\tau_2$  .
- The survival of both until time  $t > 0$  . By (5) and  $Y < y^*$  has the probability  $\int_0^{y^*} f_Y(y) e^{[-y(\lambda_1 T_1 + \lambda_2 T_2)]} dy$  and the unconditional probability of the survival event is  $E[e^{[-y(\lambda_1 T_1 + \lambda_2 T_2)]}]$
- so by Bayes rule , the conditional probability of  $Y < y^*$

$$P[Y \leq y^*] = \int_0^{y^*} f_Y(y) \frac{e^{[-y(\lambda_1 T_1 + \lambda_2 T_2)]}}{E[e^{[-y(\lambda_1 T_1 + \lambda_2 T_2)]}]} dy \quad (11)$$

- The conditional density of  $f_{Y,F_i}$  of  $Y$  conditional on the survival of both obligors is given by the exponential transformation of  $Y$  . The likelihood ratio of the transformation is

$$\frac{f_{Y,F_i}(y)}{f_Y(y)} = \frac{e^{[-y(\lambda_1 T_1 + \lambda_2 T_2)]}}{E[e^{[-y(\lambda_1 T_1 + \lambda_2 T_2)]}]} \quad (12)$$

# The Frailty

- The likelihood ratio has the effect of adjusting the probability density of Y as time goes by and as long as no defaults happen there will be more mass for low Y
- The resulting conditional survival probabilities and default intensities are :

$$P[\tau_1 > T_1, \tau_2 > T_2 | F_t] = \frac{E[e^{-T(\lambda_1 T_1 + \lambda_2 T_2)}]}{E[e^{-Y(\lambda_1 t + \lambda_2 t)}]} \quad (13)$$

$$h_i(t) = \lambda_i \frac{E[e^{-T(\lambda_1 T_1 + \lambda_2 T_2)}]}{E[e^{-Y(\lambda_1 t + \lambda_2 t)}]} \quad (14)$$

- Using the Laplace transform of Y and its derivative and plot the hazard rate as a function of time we will see a decreasing behaviour : No defaults are good news for all the rest of the obligors . In a  $\Gamma(\alpha, \beta)$  distributed frailty we have

$$h_i(t) = \lambda_i \frac{\alpha \beta}{1 + \beta(\lambda_1 t + \lambda_2 t)} \quad (15)$$

- Showing a decreasing decay with t similar to an exponential decay .
- **Let us see the case when we have a default of an obligor** . Then we need the conditional density of Y , conditional to observing a default of obligor 1 at time T1
- 2.6 with the density  $f_Y(y)$  of Y will gives us the joint density of  $(\tau_1, \tau_1, Y)$  . Then we evaluate the joint density at  $\tau_1 = T_1$  and divide it by the marginal density to get the conditional density of  $(\tau_1, Y)$  conditional on  $\tau_1 = T_1$  . The density of  $\tau_1$  can be found by differentiating 2.7 . The result is :

$$f(t_2, y | \tau_1 = T_1) = \frac{f(t_1 = T_1, t_2, y)}{\lambda_1 E[Y e^{-Y \lambda_1 T_1}]} = y \lambda_2 e^{-y \lambda_2 t_2} \frac{y e^{-y \lambda_1 T_1}}{E[Y e^{-Y(\lambda_1 T_1)}]} f_Y(y) \quad (16)$$

# The Frailty

- For  $\tau_1=T_1$  resulted in a likelihood contribution of  $\frac{ye^{-y\lambda_1 T_1}}{E[Ye^{-Y(\lambda_1 T_1)}]}$  to the joint density .
- If we now integrate out  $\tau_2=T_2$  and divide by its unconditional probability we reach the conditional density of Y conditional on  $\tau_1=T_1$  and  $\tau_2>T_2$  :

$$f(y | \tau_1 = T_1, \tau_2 > T_2) = \frac{ye^{-y\lambda_1 T_1} e^{-y\lambda_2 T_2}}{E[Ye^{-Y(\lambda_1 T_1)}] e^{-Y(\lambda_2 T_2)}} f_Y(y) \quad (17)$$

$$= \underbrace{\frac{e^{-y(\lambda_1 T_1 + \lambda_2 T_2)}}{E[e^{-Y(\lambda_1 T_1 + \lambda_2 T_2)}]}}_{\text{No default}} \cdot \underbrace{\frac{y}{E[Y | \tau_1 \geq T_1 \wedge \tau_2 \geq T_2]}}_{\text{Arrival of a default}} f_Y(y) \quad (18)$$

No default



Arrival of a default



- If we now integrate out  $\tau_2=T_2$  and divide by its unconditional probability we reach the conditional density of Y conditional on and
- In equation (18) the likelihood is decomposed as the product of two components : First , the conditioning on survival up to default  $\tau_1 \geq T_1$  and  $\tau_2 \geq T_2$  and the second factor from additionally conditioning on the default time itself  $\tau = T_1$  . The denominator is just the normalisation of these two contributions .
- If the default had not happened at  $T_1$  we would just have the contribution of the first factor . We have an additional contribution if the default happens and it is proportional to y . Surprise, Surprise .

**THE ADDITIONAL DEFAULT CONTRIBUTES TO HIGHER VALUES OF Y IN THE DISTRIBUTION**

# The Frailty

- Finally if the second obligor defaults immediately after the first one by (16) and (17) we have :

$$h_2(T_1) = \lambda_2 E[Y | \tau_1, \tau_2 > T_1] = \lambda_2 \frac{E[Y^2 e^{-sY}]}{E[Y e^{-sY}]} \quad (19)$$

- The default intensity just before the default of obligor 1 was :

$$h_2(T_1^-) = \lambda_2 E[Y | \tau_1 \geq T_1, \tau_2 > T_1] = \lambda_2 \frac{E[Y e^{-sY}]}{E[Y e^{-sY}]} \quad (20)$$

- The size of the jump ( %) in the default intensity of obligor 2 after a default of obligor 1 at  $T_1$  is :

$$\frac{h_2(T_1) - h_2(T_1^-)}{h_2(T_1^-)} = \frac{\text{Var}[Y | \tau_1 \geq T_1 \wedge \tau_2 \geq T_1]}{(E[Y | \tau_1 \geq T_1 \wedge \tau_2 \geq T_1])^2} \quad (21)$$

- The size of the relative jump is the square of the coefficient of variation of the frailty ( $\sigma_Y / \mu_Y$ )

**THE MAGNITUDE OF THE UNCERTAINTY OF Y DETERMINES THE SIZE OF THE INFORMATION BASED CONTAGION**

# The frailty . Conclusion .

- New defaults of an obligor will have an effect on the frailty towards more risky values .
- The survival of the other obligors becomes worse at the times of default
- This type of contagion is due only to the market information not changes in the creditworthiness of the obligor
- The size of the contagion is the size of the relative jump in the default hazard rate of obligor 2 upon default of obligor 1
- This relative jump is equal to the ratio of variance of  $Y$  to the square of the mean of  $Y$  . The higher the variance the higher the information that a default has and therefore a higher contagion effect .
- We found that we could separate the contagion jumps from the non-contagion part of the hazard rates .
- Similar jumps can be observed in the copula-based models of default risk . The main strength and weakness of the copula is the separation of the dependence structure from the marginal survival probabilities . It makes the calibration very easy but different default intensities with high copula dependence will exhibit very unreasonable updating properties( Not good infection properties ) .
- As we saw in the properties of the copula based models there is a strong dependence on calendar time and the distance from the initial date .

**WE HAVE BETTER MEANS TO HAVE REALISTIC LEVELS OF DEFAULT DEPENDANCE**

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**Evolutionary Portfolio Credit Management**

# More realistic modelling

- From static models like Vasicek to Gaussian Copula ( RiskMetrics )
- Gaussian copula :
  - Higher degrees of freedom
  - May want higher degrees of freedom ( t-copula in some cases )
  - Straight forward simulation
  - No analytical tractability
- Updated Dynamic Copula functions or Generalized Archimedean with the introduction of the Frailty model
  - Interesting if  $Y$  is  $\Gamma(\alpha, \beta)$  distributed ( Clayton copula )
  - In general are more tractable
  - More stable
  - Easily interpreted :  $Y$  are unobserved stress factors on the default intensities of the individual obligors
  - Contagion effects

# More realistic modelling

- The largest difference between the classical approach ( Intensity based models ) and the Frailty models is the presence of  $Y$  that influences the default intensities .
- If all default intensities were perfectly observable then the contagion should not exist and  $Y$  should be constant .
- In we could not observe joint influences on defaults of obligors then we should model dependencies with the Copula only .
- The truth is probably somewhere in the middle :
  - Some common influences are observable therefore  $\lambda_i$  should have dependent common factor  $Y$  .
  - Some are unobservable therefore modelled using a copula
  - Some are partially observable so contributions to both .

**PAYING ATTENTION TO MODEL AND PARAMETER RISKS SHOULD INCREASE THE QUALITY OF INSIGHTS**

# Conclusions

**BEFORE MAKING DECISIONS CHECK MODEL AND PARAMETER RISKS**

**CHECK UNCERTAINTY AROUND THE RISK RESULTS**

**CREDIT EXPERIENCE AND INTUITION ARE STILL THE BEST TO MINIMIZE THE ABOVE**