



*risklab germany*

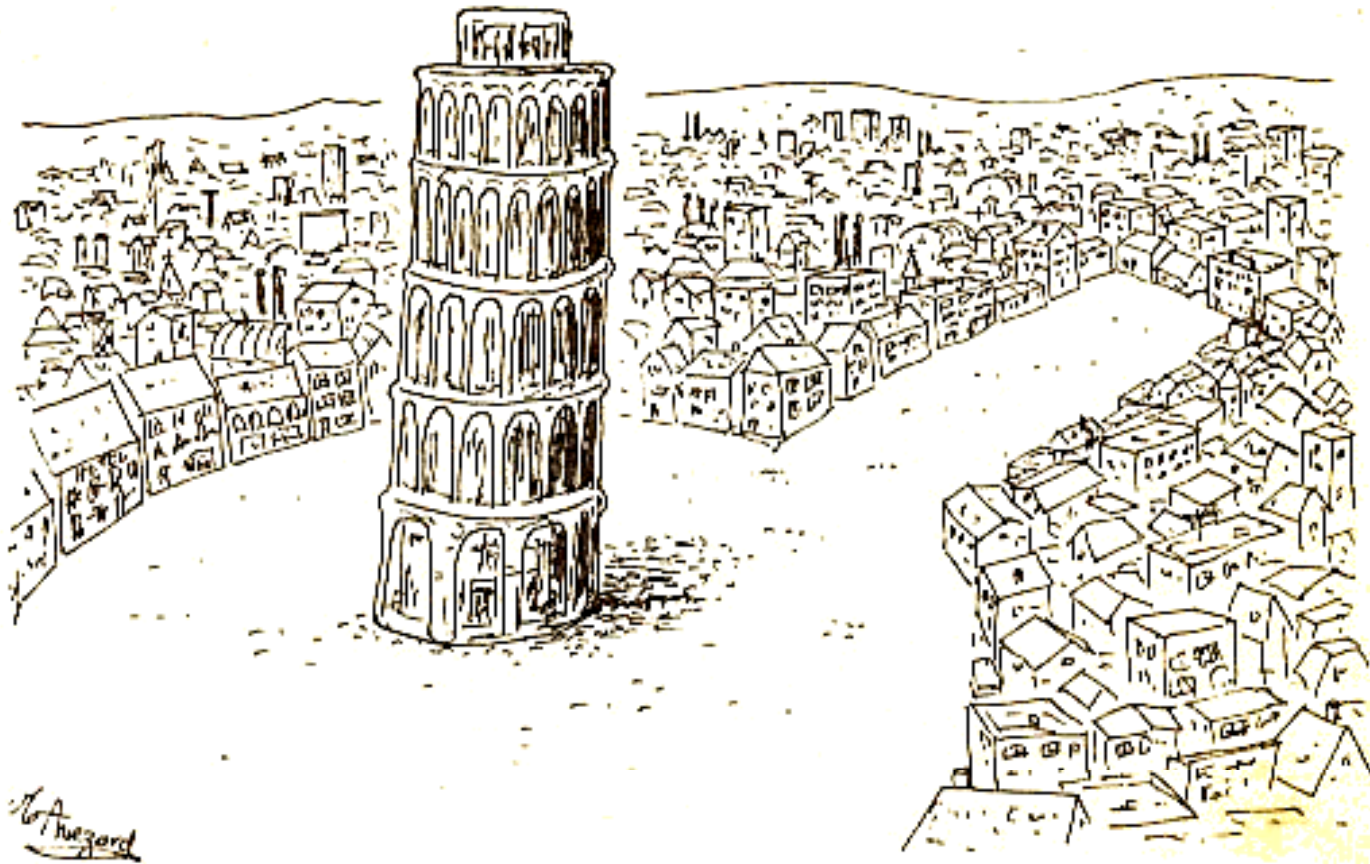
Private Research Institute for Financial Studies

# Portfolio Optimization Under Credit Risk

Prof. Dr. Rudi Zagst

Madrid, October 18, 2001

# Lessons from Pisa



Sans parois.

© ZEVAR 1976 (LE PELERIN)

# Lessons from the Market



- **Credit Information**

- Definition of Credit Risk
- Yield Curve Behaviour
- Quality Changes
- Credit Spread Behaviour

# Credit Risk

Definition according to the Dictionary of Financial Risk Management [1996]

## Credit Risk is

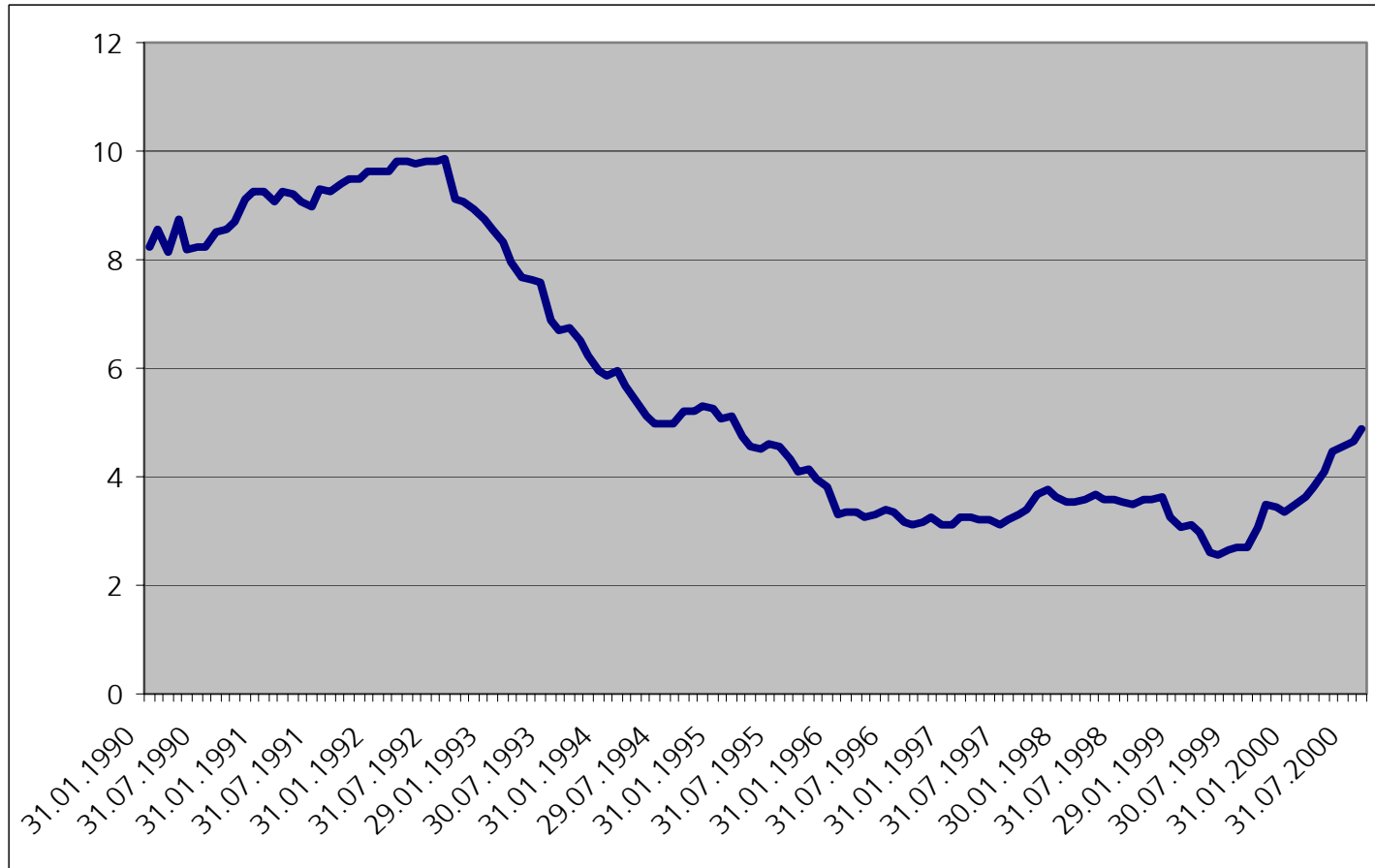
- *the exposure to loss* as a result of
  - default on a swap debt or other counterparty instrument
  - a decline in market value stemming from a credit downgrade of an issuer or counterparty
- *a component of return variability* resulting from the possibility of an event of default
- *a change in the market's perception of the probability of an event of default*, which affected the spread between two rates or reference indexes

and therefore is considered as

- *a classical downside risk*, i.e. the best case is a full repayment of the debt
- *but* may also include an *upside potential*, i.e. an increase in market value stemming from a credit upgrade of an issuer or counterparty

# Interest Rates Do Exhibit a Mean Reverting Behaviour

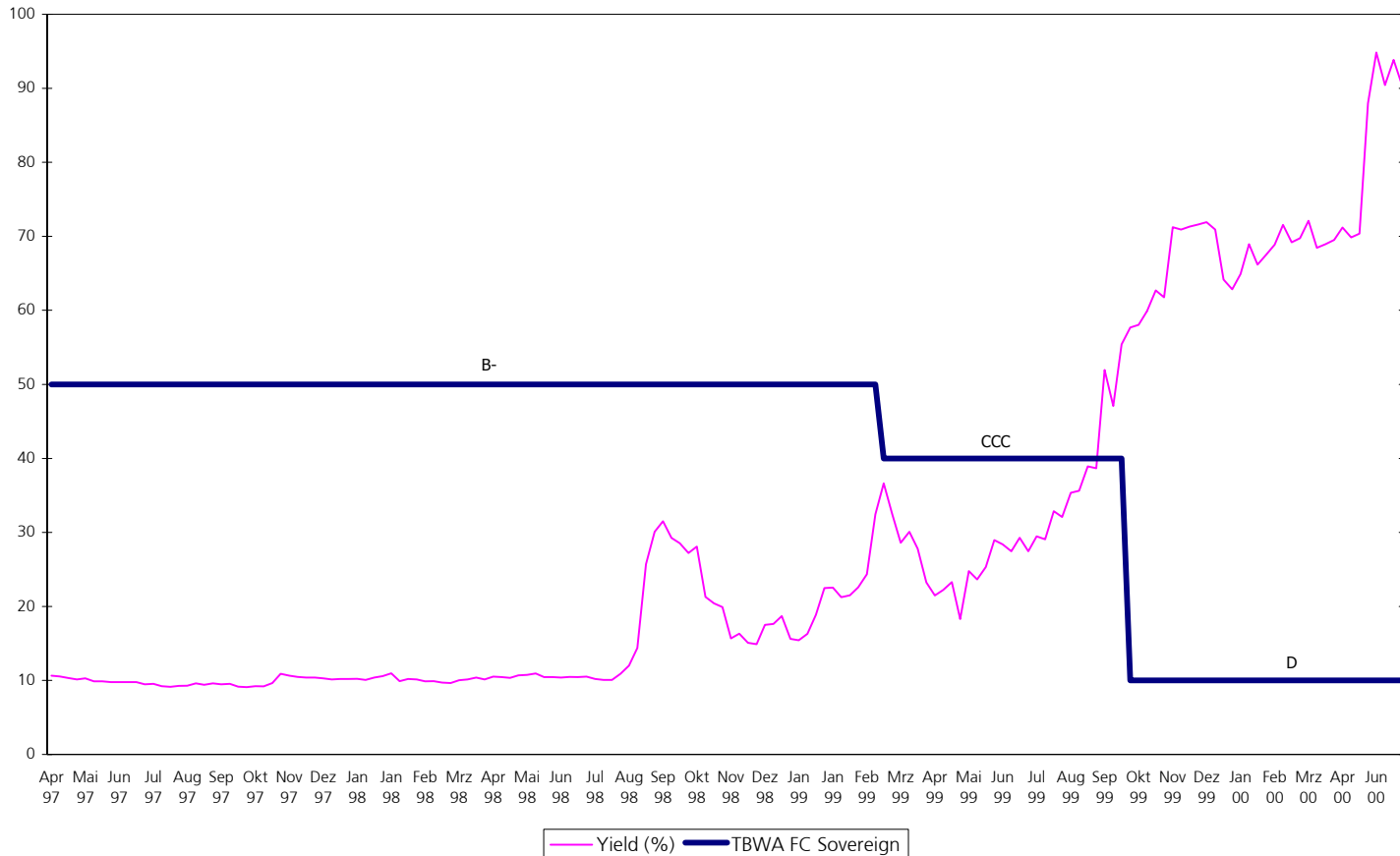
## 3-Monats Libor from 1990 until 2000 (in %)



Source: Bloomberg

# The Quality Process is Continuous

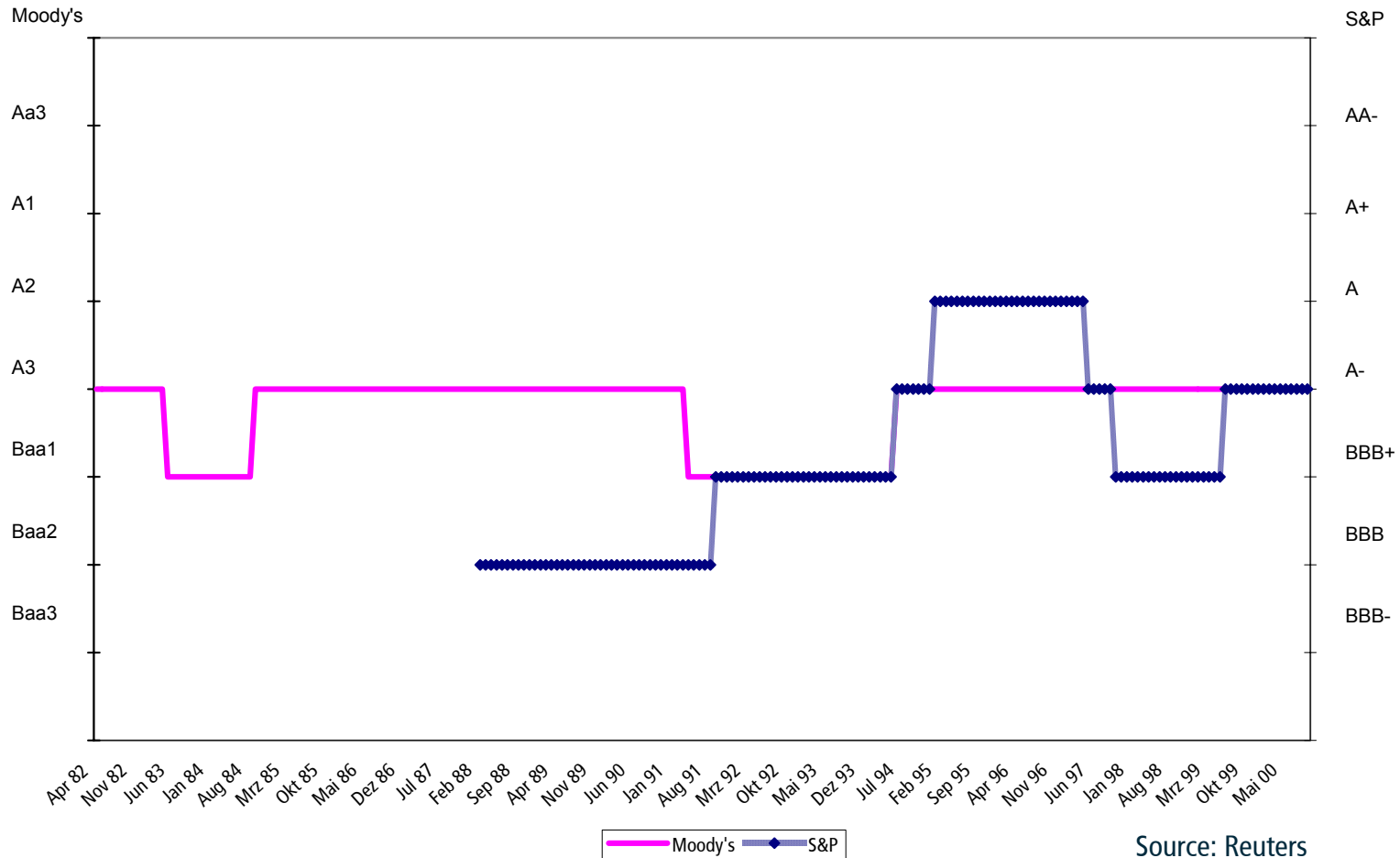
## Ecuadorian Sovereign Bond BEEC 11.25% 25/04/2002 DEM



Source: Reuters

# The Quality Process Does Exhibit a Mean Reverting Behaviour

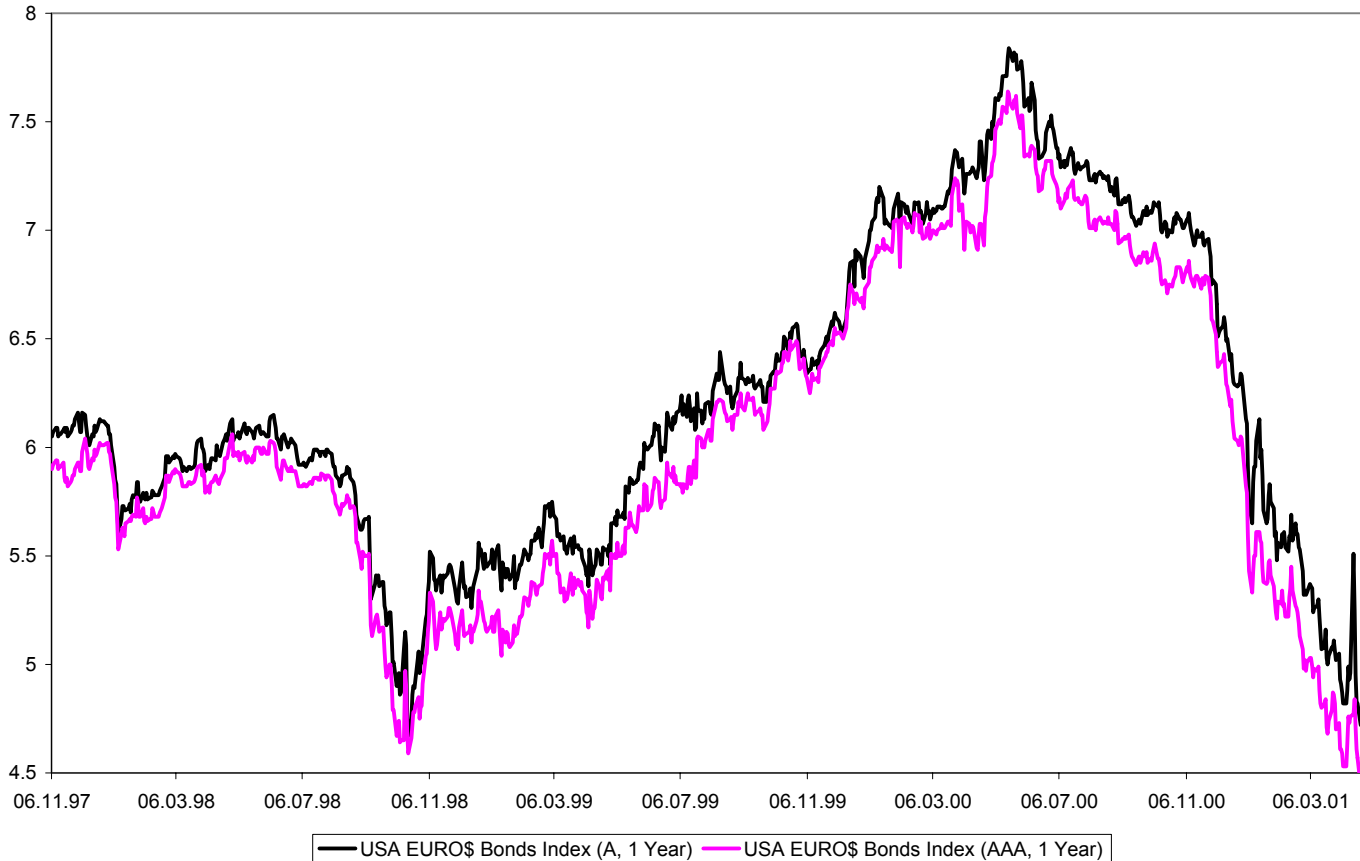
## Rating History of Hertz from April 1982 until August 2000



Source: Reuters

# The Extend of the Credit Spread Does Depend on the Rating

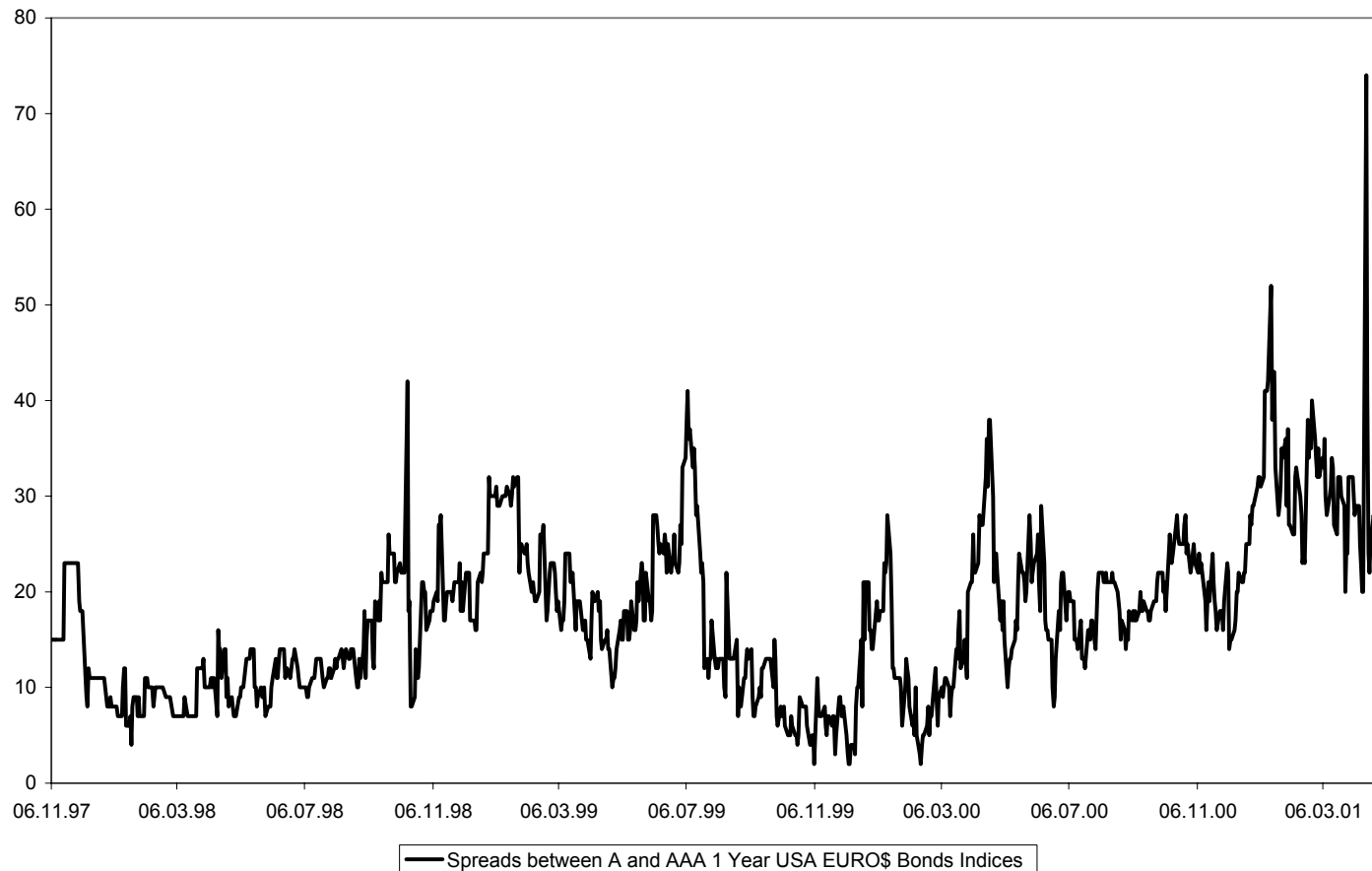
## Yields (in %) of 1 Year Euro\$ Bond Indices for the Ratings A and AAA



Source: Bloomberg

# Credit Spreads Do Exhibit a Mean Reverting Behaviour

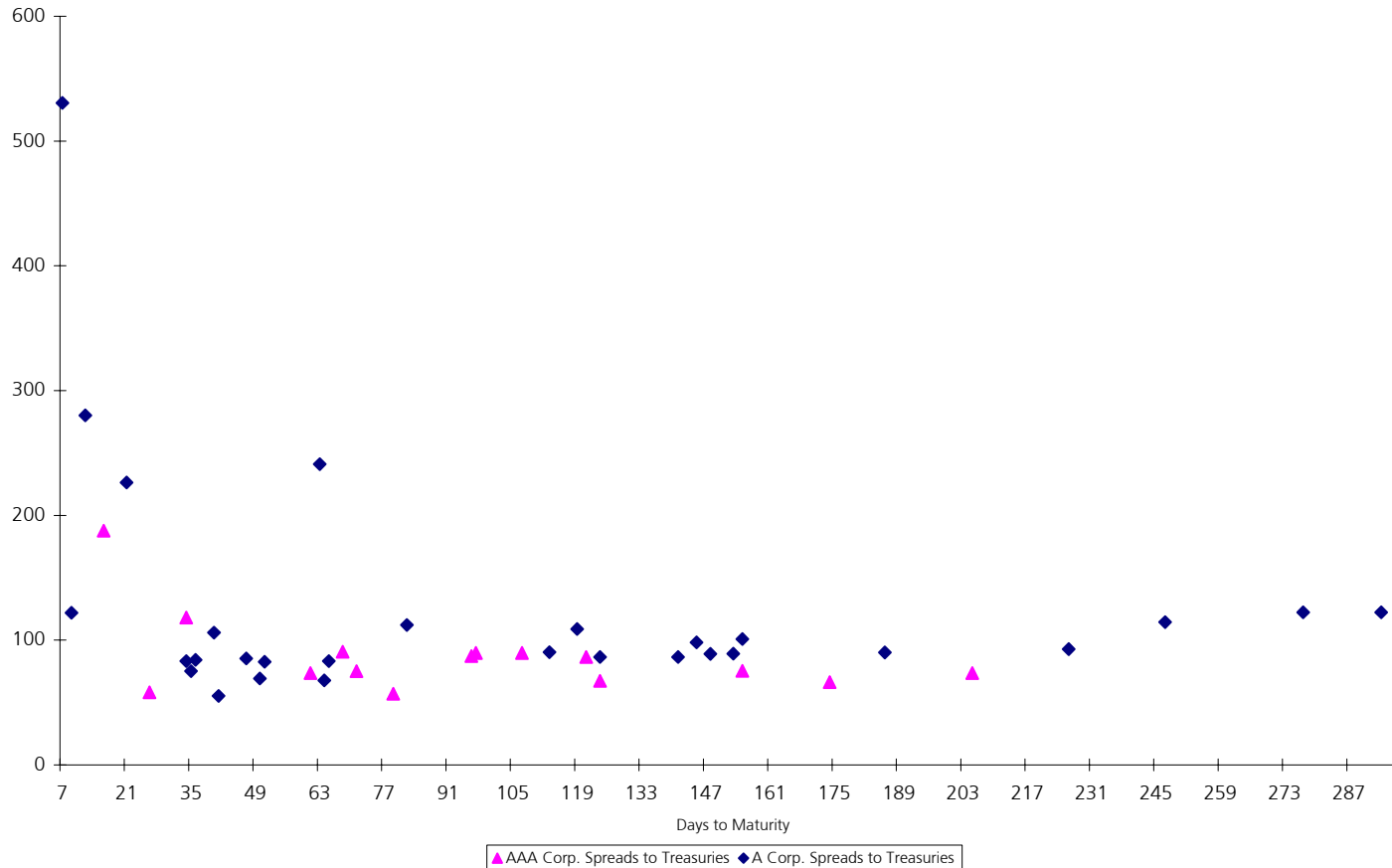
## 1 Year Credit Spread between A and AAA Euro\$ Bond Indices (in bp)



Source: Bloomberg

# Even Short-Term Credit Spreads May Be Positive

## Credit Spreads of A and AAA Rated Corporate Bonds (bp) for Different Maturities



Source: Reuters

# Models for the Pricing of Defaultable Bonds

## Overview



- **Fundamental Models**

*Merton [1974], Black und Cox [1976], Longstaff und Schwartz [1995] etc.*

- **Relative Models**

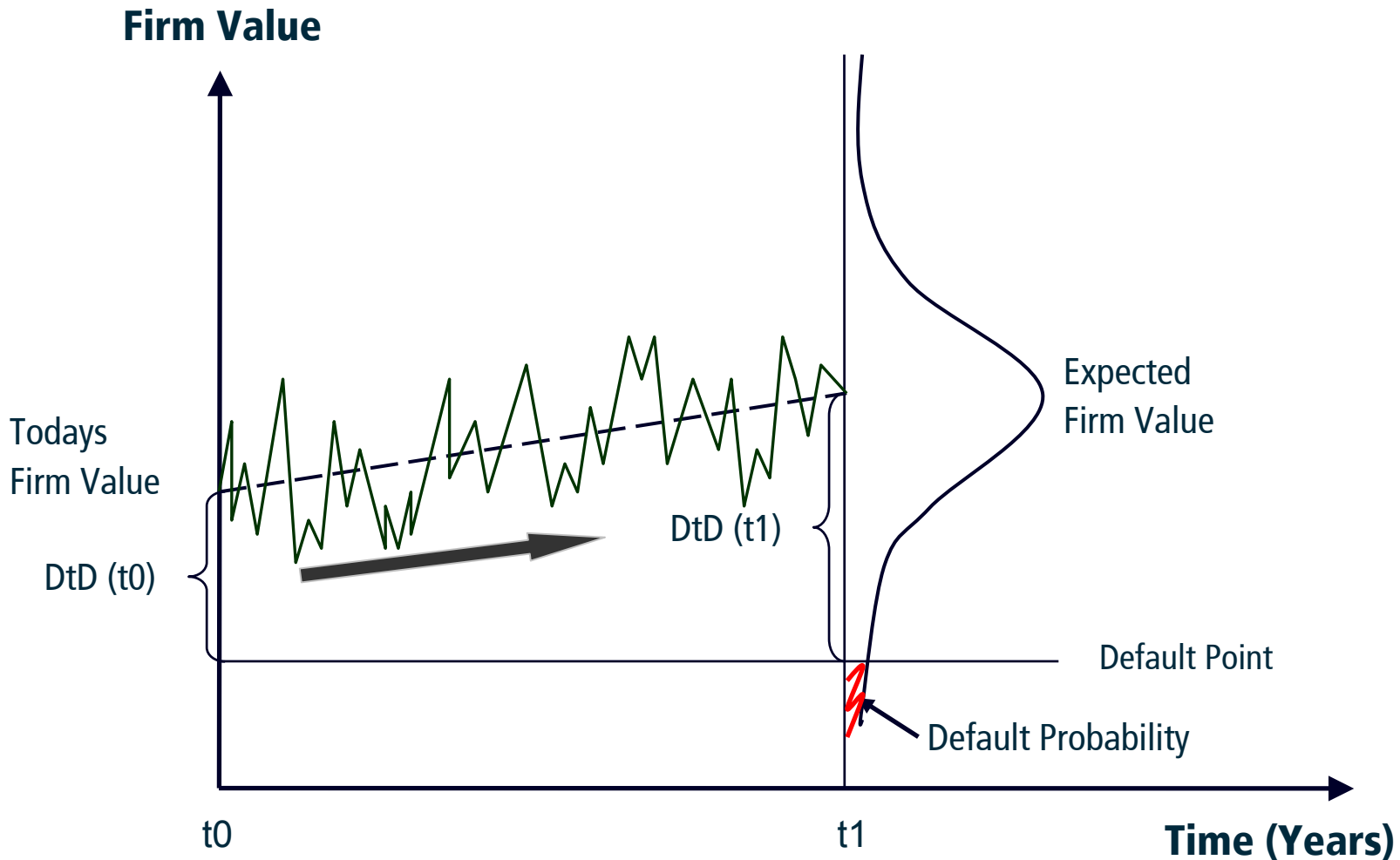
*Jarrow und Turnbull [1995], Madan und Unal [1994, 1998], Duffie und Singleton [1994-1999] etc.*

- **Hybride Models**

*Cathcart und El-Jahel [1998], Schmid und Zagst [2000] etc.*

# Fundamental Models

## Modeling of the Default Probability



# Fundamental Models

## The Merton Model (1974)



Modeling of the non-defaultable short rate  $r$ , the firm value  $V$ , and the default boundary  $D$ , i.e. default probability and recovery rate are implicitly given

- Firm finances its activities via stocks and discount bonds at a notional  $D$  and maturity  $T$
- Firm value  $V$  changes according to the SDE

$$dV_t = rV_t dt + \sigma V_t dW_t$$

with a standardized Brownian motion  $W$  and constant interest rate  $r$  and volatility  $\sigma$

- At maturity  $T$  the creditors receive  $\min(V_T, D) = D - \max(0, D - V_T)$

Hence, the value of the defaultable bond is given by

$$P^d(r, V, t, T) = D \cdot P(r, t, T) - Put(V, t, D, \sigma, r, T - t),$$

where  $Put$  is the Black-Scholes value of a European put option

# Models for the Pricing of Defaultable Bonds

## Overview



- **Fundamental Models**

*Merton [1974], Black und Cox [1976], Longstaff und Schwartz [1995] etc.*

- **Relative Models**

*Jarrow und Turnbull [1995], Madan und Unal [1994, 1998], Duffie und Singleton [1994-1999] etc.*

- **Hybride Models**

*Cathcart und El-Jahel [1998], Schmid und Zagst [2000] etc.*

## Relative Models

### Modeling of the Default Probability



Exogenous modeling of the non-defaultable short rate  $r$ , the default probability  $q$  under the equivalent martingale measure, and the recovery rate  $y$

#### Jarrow/Turnbull [1995]:

Default = first jump of a Poisson process with constant intensity parameter (hazard rate)  $h > 0$ , i.e. the probability of a default in the interval  $(t, T]$  is given by

$$q(t, T) = 1 - \exp[-h \cdot (T - t)]$$

#### Madan/Unal [1998]:

The intensity parameter is modeled by a stochastic process  $(h(t))_{t \geq 0}$ , i.e. the probability of a default in the interval  $(t, T]$  is given by

$$q(t, T) = 1 - E_Q \left[ \exp \left( - \int_t^T h(l) dl \right) \middle| \mathfrak{F}_t \right]$$

## Relative Models

### Pricing Defaultable Bonds

The value of a defaultable discount bond at time  $t$  maturing at time  $T \geq t$  is given by

$$P^d(r, h, t, T) = E_Q \left( \frac{B(t)}{B(T)} \cdot X(T) \mid \mathfrak{F}_t \right)$$

where

- $Q$  is the martingale measure on the complete filtered probability space  $(\Omega, \mathfrak{F}, Q, (\mathfrak{F}_t)_{t \in [0, T^*]})$
- $B(t)$  is the value of the money market account at time  $t$
- $X(T)$  is the value of the payoff of the bond at time  $T$

Under appropriate independence assumptions we get:

$$\begin{aligned} P^d(r, h, t, T) &= E_Q \left[ \frac{B(t)}{B(T)} \cdot X(T) \mid \mathfrak{F}_t \right] = P(r, t, T) \cdot E_Q [X(T) \mid \mathfrak{F}_t] \\ &= P(r, t, T) \cdot [1 - q(t, T) + y \cdot q(t, T)] \end{aligned}$$

where  $y$  denotes the recovery rate at time  $T$

# Models for the Pricing of Defaultable Bonds

## Overview



- **Fundamental Models**

*Merton [1974], Black und Cox [1976], Longstaff und Schwartz [1995] etc.*

- **Relative Models**

*Jarrow und Turnbull [1995], Madan und Unal [1994, 1998], Duffie und Singleton [1994-1999] etc.*

- **Hybride Models**

*Cathcart und El-Jahel [1998], Schmid und Zagst [2000] etc.*

## The Model of Schmid and Zagst [2000]

### Modeling of the Stochastic Processes under the Martingale Measure Q

- **Dynamic of the yield curve (non-defaultable short rate)**

$$dr(t) = [\theta_r(t) - \hat{a}_r \cdot r(t)]dt + \sigma_r d\hat{W}_r(t), t \in [0, T^*], \hat{a}_r > 0, \sigma_r > 0$$

- **Dynamic of the counterparty quality (uncertainty index)**

$$du(t) = [\theta_u - \hat{a}_u \cdot u(t)]dt + \sigma_u \sqrt{u(t)} d\hat{W}_u(t), t \in [0, T^*], \theta_u \geq 0, \hat{a}_u > 0, \sigma_u > 0$$

- **Dynamic of the yield spreads (short rate spreads)**

$$ds(t) = [b_s \cdot u(t) - \hat{a}_s \cdot s(t)]dt + \sigma_s \sqrt{s(t)} d\hat{W}_s(t), t \in [0, T^*], b_s > 0, \hat{a}_s > 0, \sigma_s > 0$$

- **The Wiener processes  $\hat{W}_r$ ,  $\hat{W}_u$ , and  $\hat{W}_s$  are uncorrelated\***

\* assumption can be relaxed

# The Model of Schmid and Zagst [2000]

## Evaluation of Non-Defaultable Bonds

### Theorem 1 (Hull and White [1990]).

The time  $t$  value  $P(t,T)$  of a non-defaultable discount bond with maturity  $T \geq t$  is given by

$$P(t, T) = A(t, T) \cdot e^{-B(t, T) \cdot r}$$

with

$$B(t, T) = \frac{1}{\hat{a}_r} \cdot \left(1 - e^{-\hat{a}_r \cdot (T-t)}\right)$$

and

$$\ln A(t, T) = \ln\left(\frac{P(0, T)}{P(0, t)}\right) - B(t, T) \cdot \frac{\partial \ln P(0, t)}{\partial t} - \frac{\sigma_r^2}{4\hat{a}_r^3} \cdot \left(e^{-\hat{a}_r \cdot T} - e^{-\hat{a}_r \cdot t}\right)^2 \cdot \left(e^{2 \cdot \hat{a}_r \cdot t} - 1\right).$$

# The Model of Schmid and Zagst [2000]

## Evaluation of Defaultable Bonds

### Theorem 2 (Defaultable Discount Bond, Schmid und Zagst [2000]).

The time  $t$  value  $P^d(t,T)$  of a defaultable discount bond with maturity  $T \geq t$  is given by

$$P^d(t,T) = A^d(t,T) \cdot e^{-B(t,T) \cdot r - C(t,T) \cdot s - D(t,T) \cdot u}$$

with

$$C(t,T) = \frac{1 - e^{-\delta_s \cdot (T-t)}}{K_1^{(s)} - K_2^{(s)} \cdot e^{-\delta_s \cdot (T-t)}}$$

$$\text{with } \delta_s = \sqrt{\hat{a}_s^2 + 2 \cdot \sigma_s^2}$$

$$\text{and } K_k^{(s)} = \frac{1}{2} \cdot (\hat{a}_s - (-1)^k \cdot \delta_s), \quad k \in \{1,2\}$$

$$D(t,T) = -\frac{2 \cdot v'(t,T)}{\sigma_u^2 \cdot v(t,T)}$$

with  $v(t,T)$  complicated

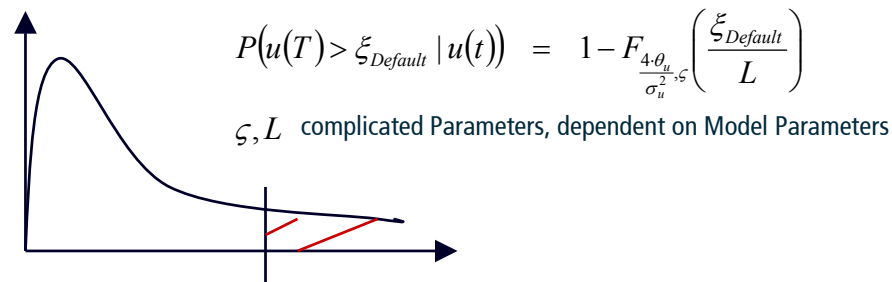
$$A^d(t,T) = A(t,T) \cdot e^{-S(t,T)}$$

$$\text{with } S(t,T) = -\frac{2 \cdot \theta_u}{\sigma_u^2} \cdot \ln \left| \frac{v(T,T)}{v(t,T)} \right|$$

# Hybride Models as Generalization of Fundamental and Relative Models

## Fundamental Models:

- Potential consideration of structural information (firm value)
- Makroeconomic factors as explaining factors for the quality of the debtor
- Definition of default



## Relative Models:

- Short rate credit spread can be expressed using the intensity and recovery rate process
- $q(t, T) = 1 - E_Q \left[ \exp \left( - \int_t^T h(l) dl \right) | \mathfrak{F}_t \right]$  VS.  $q(t, T) = 1 - F_{\frac{4\theta_u}{\sigma_u^2}, \varsigma} \left( \frac{\xi_{Default}}{L} \right)$

# Model Comparison

## Fundamental, Relative, and Hybride Models

|   | <b>Fundamental Models</b> | <b>Relative Models</b> | <b>Hybride Models</b> |
|---|---------------------------|------------------------|-----------------------|
| <b>Complexity</b>                         | --                        | +                      | +                     |
| <b>Market- and Structural Information</b> | +                         | -                      | ++                    |
| <b>Model Performance</b>                  | -                         | +                      | +                     |
| <b>Consideration of Ratings</b>           | -                         | ++                     | ++                    |
| <b>Pricing of Credit Derivatives</b>      | --                        | ++                     | ++                    |

# The Model of Schmid and Zagst [2000]

## Results and Applications

- Cox-Ingersoll-Ross model to describe the short rate
- Numerical procedure for the pricing of credit derivatives allowing for correlations of the Wiener processes
- *Empirical parameter estimation using yield and spread curve data*
- Implicite parameter estimation using cap prices, yield curve, and rating information
- *Simulation, risk and portfolio management*

# The Model of Schmid and Zagst [2000]

## Empirical Parameter Estimation

- 33 AAA-rated German discount, 36 AA-rated Italian discount, and 38 (A-)-rated Greek coupon government bonds
- Maturities of the bonds between 0.36 and 20.8 years
- 22 month time series of Reuters daily bond prices from November 1, 1999 until October 23, 2000
- All prices are denominated in Euros
- German bonds are considered to be free of default risk
- Parameter estimation using the Kalman filter

# The Model of Schmid and Zagst [2000]

## Parameter Estimates

| Parameter      | Estimation<br>Germany |
|----------------|-----------------------|
| $\theta_r$ (%) | 1.4413                |
| $a_r$          | 0.2382                |
| $\sigma_r$ (%) | 1.5581                |
| $\lambda_r$    | -0.0861               |
| $r(0)$ (%)     | 4.2434                |

$$\hat{a}_x = a_x + \lambda_x \cdot \sigma_x^2, \quad x \in \{r, u, s\}$$

| Parameter      | Estimation<br>Italy | Estimation<br>Greece |
|----------------|---------------------|----------------------|
| $b_s$          | 0.2748              | 0.3432               |
| $a_s$          | 0.0477              | 0.1678               |
| $\sigma_s$ (%) | 15.832              | 44.6885              |
| $\lambda_s$    | -1.8987             | -0.2159              |
| $s(0)$ (%)     | 0.1296              | 0.2325               |
| $\theta_u$ (%) | 0.0031              | 0.0392               |
| $a_u$          | 0.0687              | 0.0750               |
| $\sigma_u$ (%) | 3.0482              | 6.7231               |
| $\lambda_u$    | -1.2281             | -1.1091              |
| $u(0)$ (%)     | 0.5112              | 1.2704               |

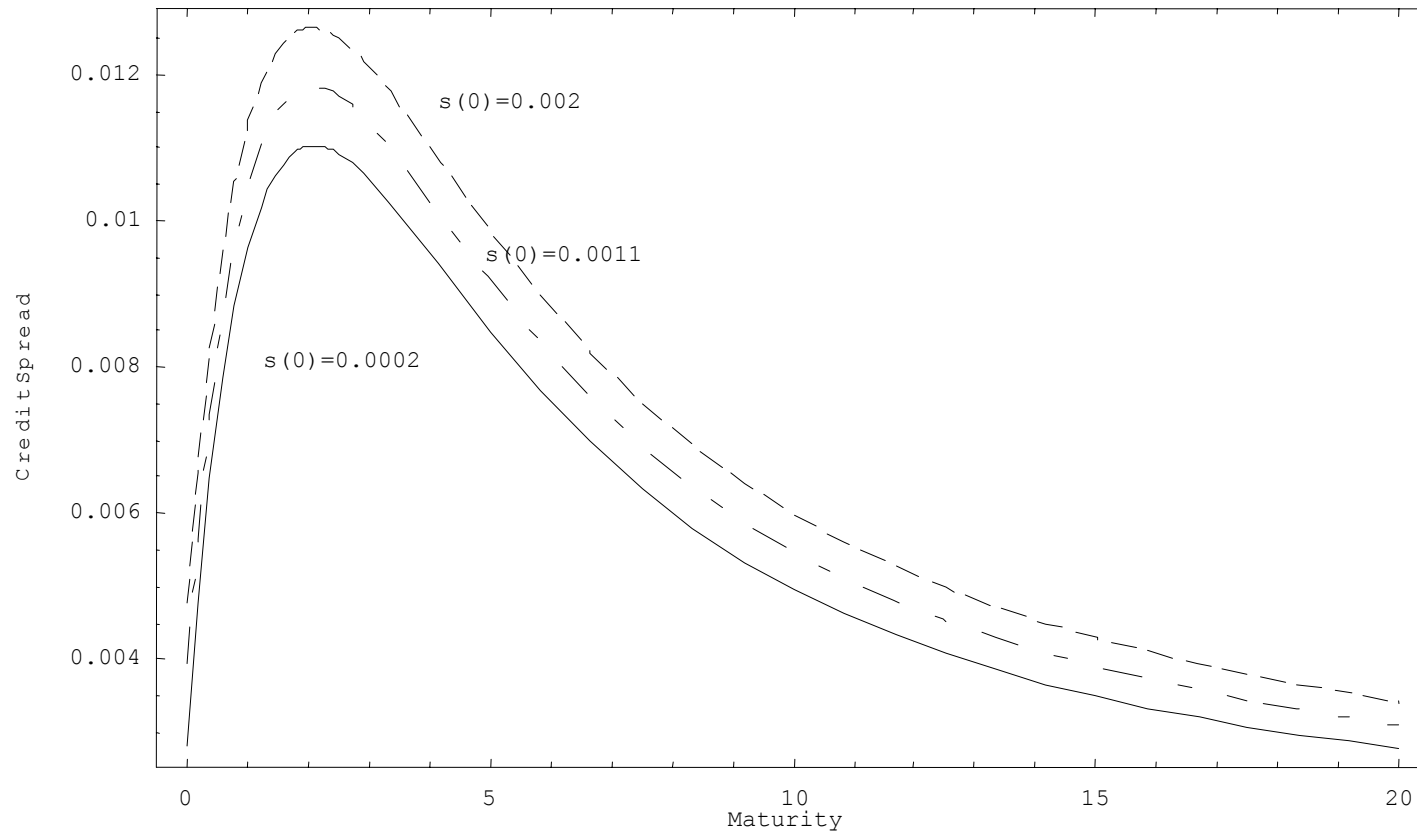
# The Model of Schmid and Zagst [2000]

## Statistical Tests

- ✓ Test on *Normal distribution* of the standardized difference between the new observations and the corresponding Kalman filter forecasts (standardized Kalman filter innovations) as proposed in Greene [1993], p. 310 (see Harvey [1989], p. 260, for a similar test)
- ✓ Test if the filtered values for the Wiener processes are *uncorrelated* as proposed in Larsen and Morris [2001], p. 626
- ✓ Box-Ljung tests on *serial correlation* of the standardized Kalman filter innovations (Q-statistic)
- ✓ Tests on *homoscedasticity* of the standardized Kalman filter innovations (H-statistic)
- ✓ *In-sample* tests on the *model performance* (linear regression)
- ✓ *Out-of-sample* tests on the *model performance* (linear regression)

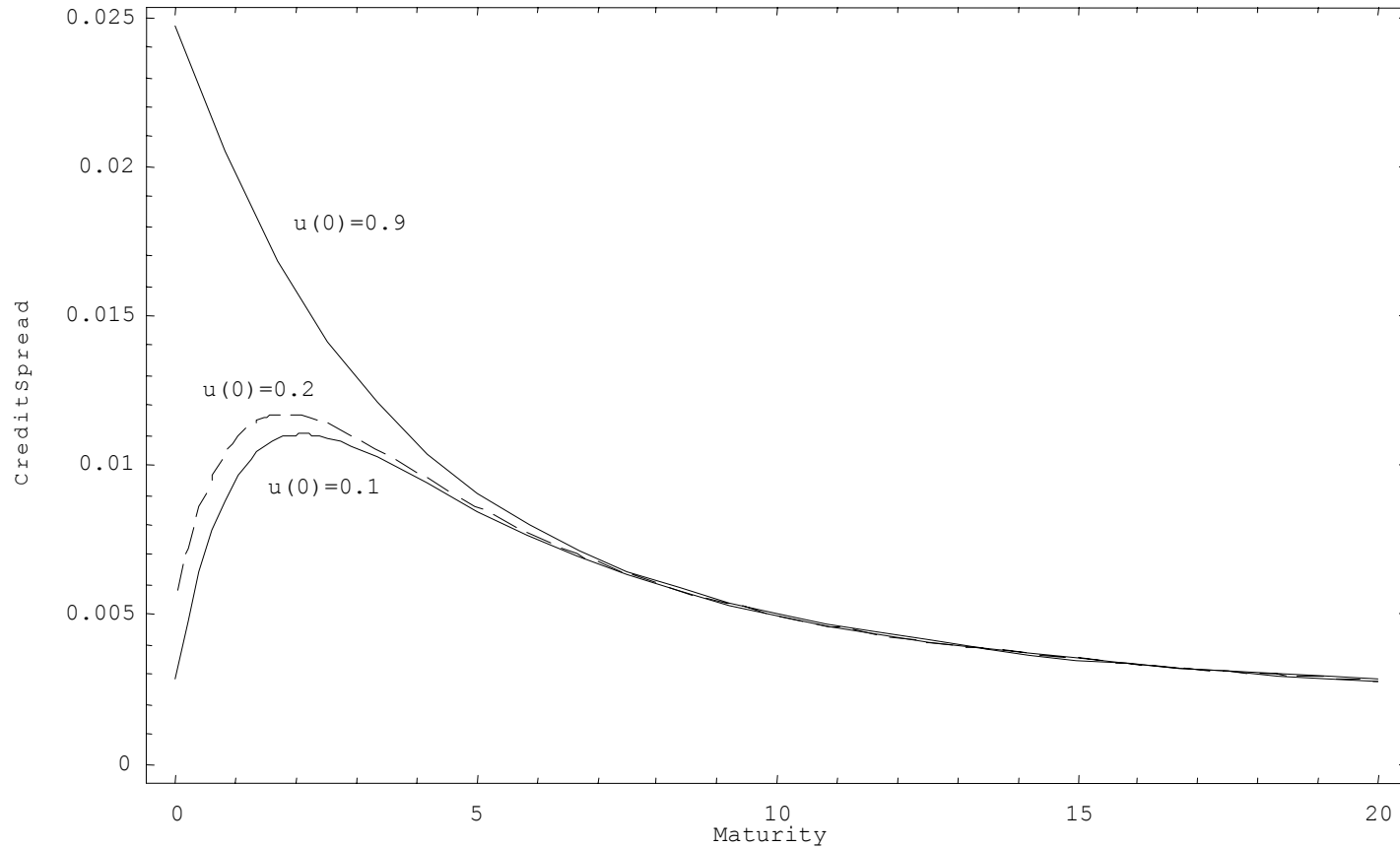
# The Model of Schmid and Zagst [2000]

## Illustrations: Structure of the Model Implied Credit Spreads



# The Model of Schmid and Zagst [2000]

## Illustrations: Structure of the Model Implied Credit Spreads



# Portfolio Optimization under Shortfall and Liability Constraints



- **Credit Portfolio Management**

- Definitions
- Simulation
- Downside Risk
- Case Study

# Credit Portfolio Management

## Definitions

- Planning horizon  $T \in [0, T^*]$
- $P_1(t), P_2(t), \dots, P_n(t)$  prices of sovereign bonds  $i=1, \dots, n$  at time  $t \in [0, T]$
- $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$  portfolio of the sovereign bonds  $i=1, \dots, n$
- $B(t)$  benchmark value at time  $t \in [0, T]$
- $T = \{T_1, T_2, \dots, T_L\} \subset [0, T]$  liability payment dates
- $L(t)$  liability payment at time  $t \in T$
- Coupon payments between the payment dates are put on a cash account

# Credit Portfolio Management

## Simulation

- $F=(F_1, F_2, \dots, F_m)=(r, s^{(l)}, u^{(l)}, s^{(G)}, u^{(G)})$  vector of risk factors with simulations  $F^k$  and probability  $p_k > 0, k=1, \dots, K$
- $V_i(t)=V_i(F, t), i=1, \dots, n$ , value of the stochastic variable  $V_i$  (cash account or bond plus potential coupon payments between times 0 and  $t$ ) at time  $t \in [0, T]$  with simulations  $V_i^k(t)=V_i(F^k, t), k=1, \dots, K, i=1, \dots, n$
- (Simulated) future value  $V^k(\varphi, t)$  of portfolio  $\varphi=(\varphi_1, \varphi_2, \dots, \varphi_n)$  at time  $t \in [0, T]$ :

$$V^k(\varphi, t) = \sum_{i=1}^n \varphi_i \cdot V_i^{(k)}(t), \quad k = 1, \dots, K$$

# Credit Portfolio Management

## Downside Risk

- Lower Partial Moment (LPM) of order  $l \in \mathbb{N}$  corresponding to the benchmark  $B(t)$ :

$$LPM_l(\varphi, V, B, t) = \sum_{\substack{k=1 \\ V^k(\varphi, t) < B(t)}}^n p_k \cdot (B(t) - V^{(k)}(\varphi, t))^l$$

- $l=0$  shortfall probability
- $l=1$  (expected) regret
- **Goal:** Maximize final expected portfolio value (return)  
under  
Allocation constraints, limited downside risk, covered liability stream

# Credit Portfolio Management

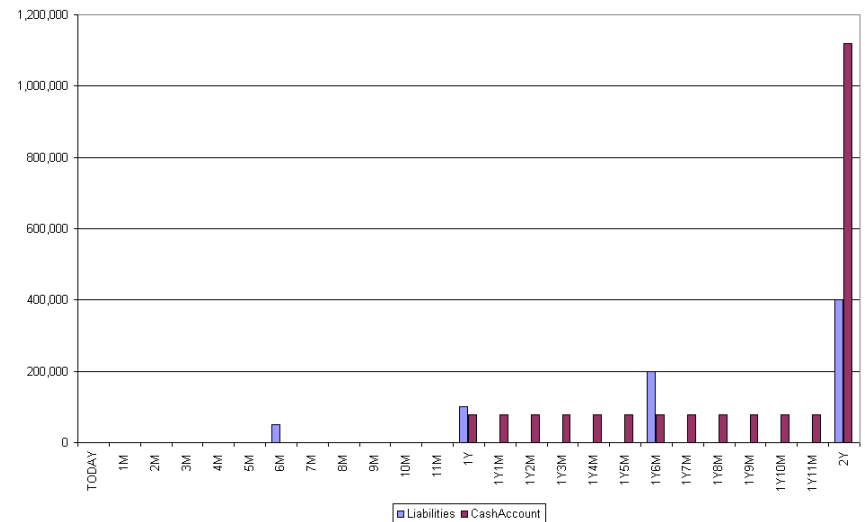
## Case Study

- Planning horizon 2 years (2Y)
- Liability payment dates  $T=\{6M,1Y,1.5Y,2Y\}$
- Liability stream (50000,100000,200000,400000) Euro
- Budget = benchmark = 1 Mio. Euro
- No short sales
- Maturity of the bonds semiannual between 1 and 3 years
- Coupons of the German, Italian, and Greek bonds are 6%,7%, and 8%
- Coupon payments between the payment dates are put on a cash account

# Credit Portfolio Management

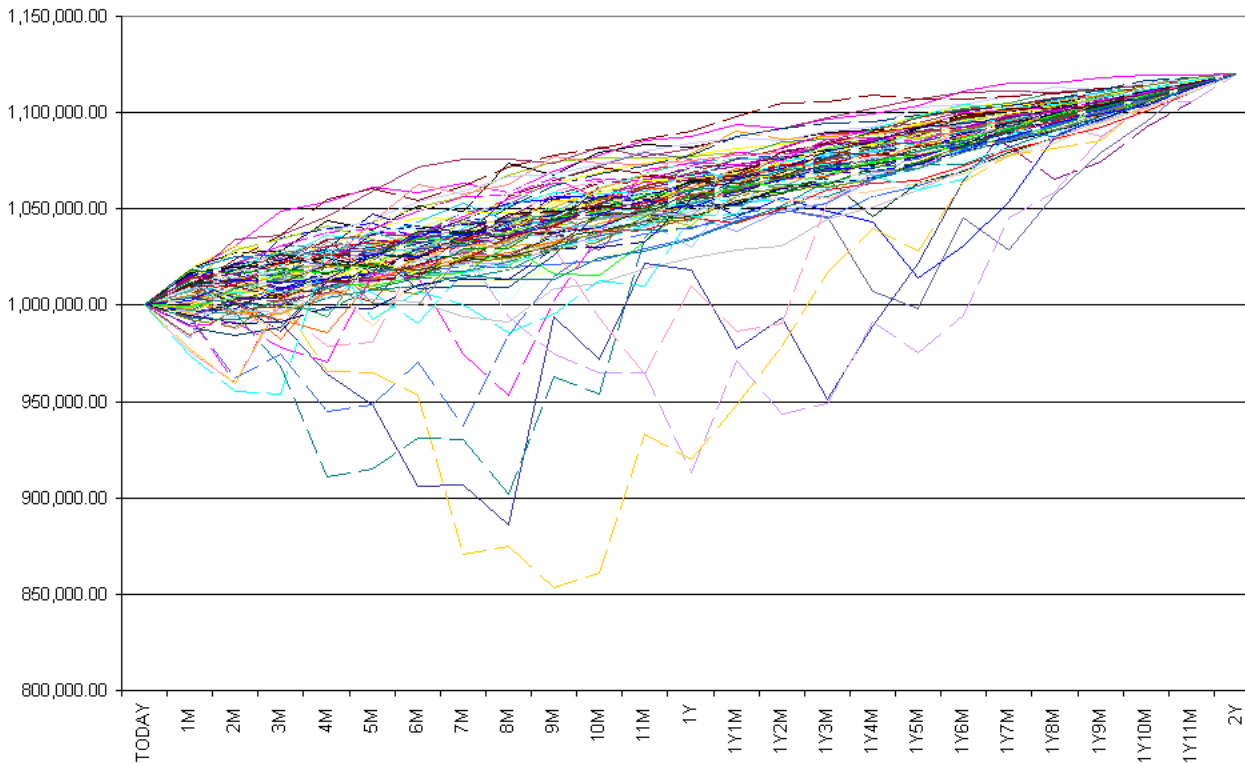
## Example 1: No Shortfall and Liability Constraints

- Optimal portfolio:  
9,658 (1 Mio. Euro ) of the 2-year 8% Greek bond
- Expected final value:  
1,120,337 Euro corresponding to an expected rate of return of 5.68%
- Probabilities of falling below the benchmark at times  $t \in T$ :  
(5%,2%,1%,0%)
- Liability stream is not covered by this portfolio:



# Credit Portfolio Management

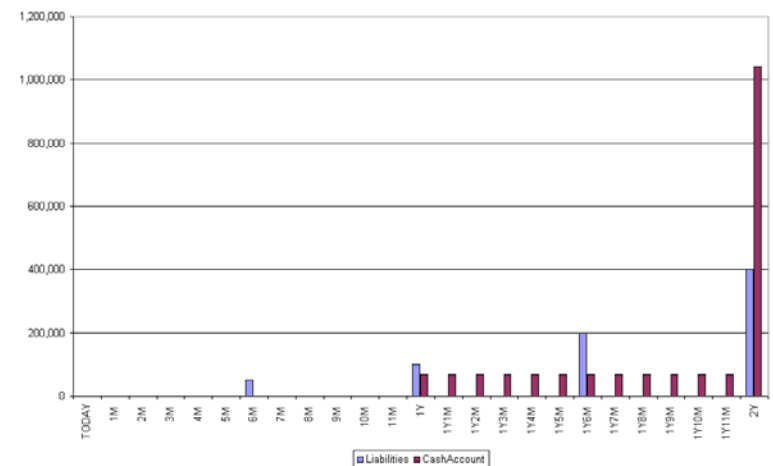
## Example 1: Simulation of the Portfolio Value (including Coupon Payments)



# Credit Portfolio Management

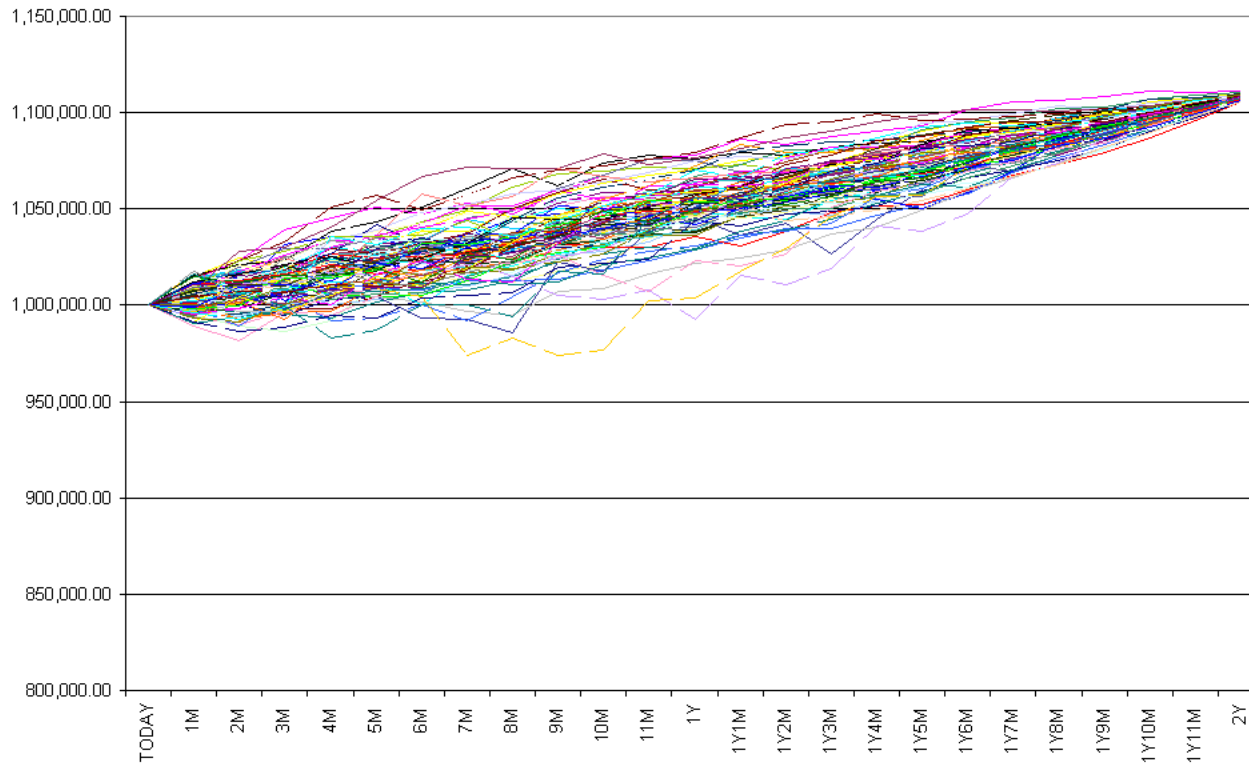
## Example 2: Additional Shortfall Constraint, No Liability Constraint

- Shortfall probability 1%
- Optimal portfolio:
  - 3,008 (311,428 Euro) of the 2-year 8% Greek bond
  - 2,485 (253,287 Euro) of the 2-year 7% Italian bond
  - 3,604 (369,108 Euro) of the 2-year 6% German bond
  - 641 (66,177 Euro) of the 3-year 6% German bond
- Expected final value:
  - 1,105,841 Euro corresponding to an expected rate of return of 5.03%
- Probabilities of falling below the benchmark at times  $t \in T$ : (1%,1%,0%,1%)
- Liability stream is not covered by this portfolio:



# Credit Portfolio Management

## Example 2: Simulation of the Portfolio Value (including Coupon Payments)



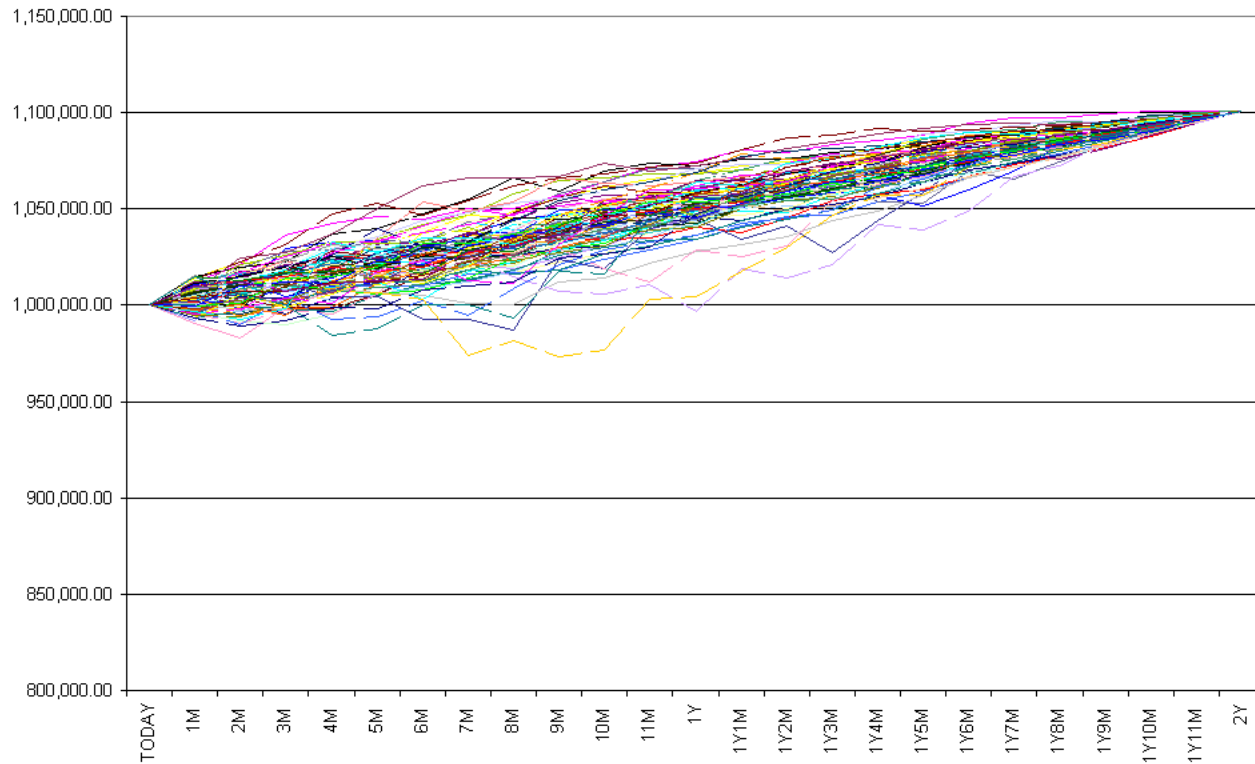
# Credit Portfolio Management

## Example 3: Additional Shortfall and Liability Constraint

- Shortfall and liability constraint at a probability of 1%
- Optimal portfolio:
  - 3,050 (315,815 Euro) of the 2-year 8% Greek bond
  - 2,133 (217,370 Euro) of the 2-year 7% Italian bond
  - 1,869 (196,040 Euro) of the 1.5-year 7% Italian bond
  - 465 ( 46,900 Euro) of the 1-year 7% Italian bond
  - 1,825 (186,959 Euro) of the 2-year 6% German bond
  - 36,916 Euro cash
- Expected final value:
  - 1,099,185 Euro corresponding to an expected rate of return of 4.73%
- Probabilities of falling below the benchmark at times  $t \in T$ : (1%,1%,0%,0%)

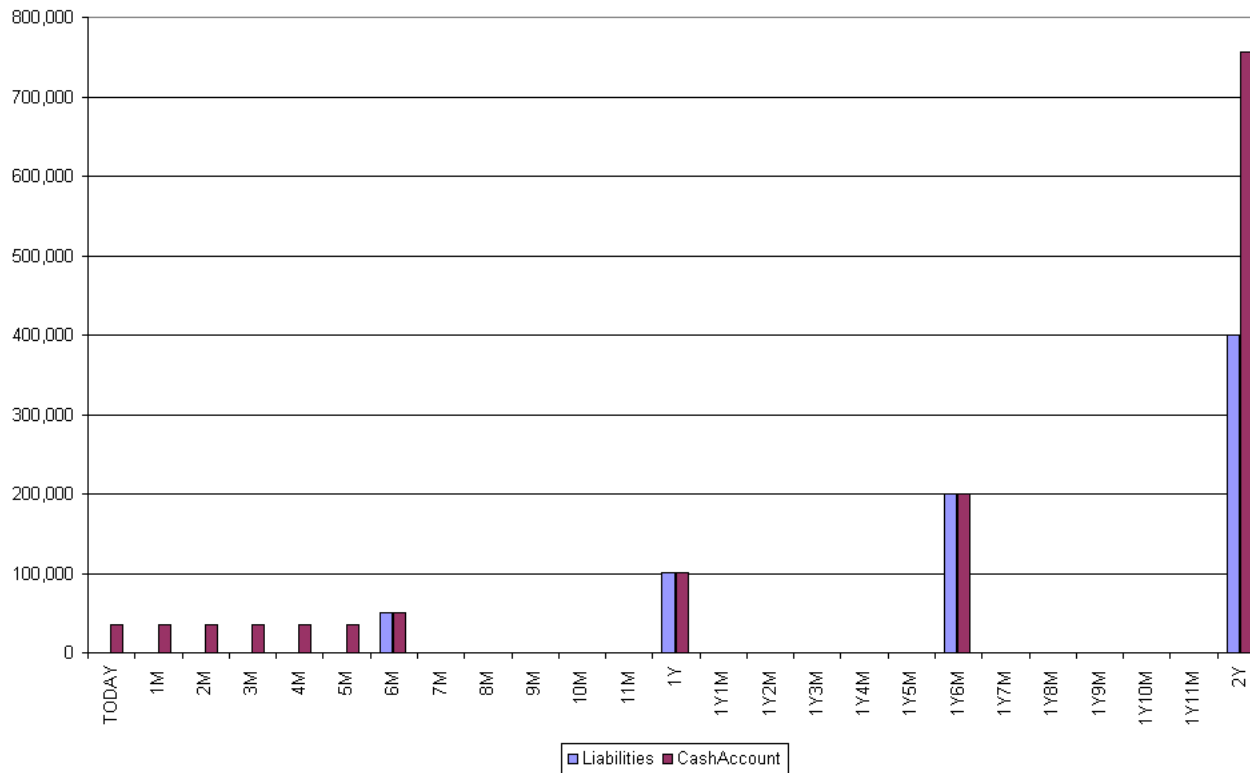
# Credit Portfolio Management

## Example 3: Simulation of the Portfolio Value (including Coupon Payments)

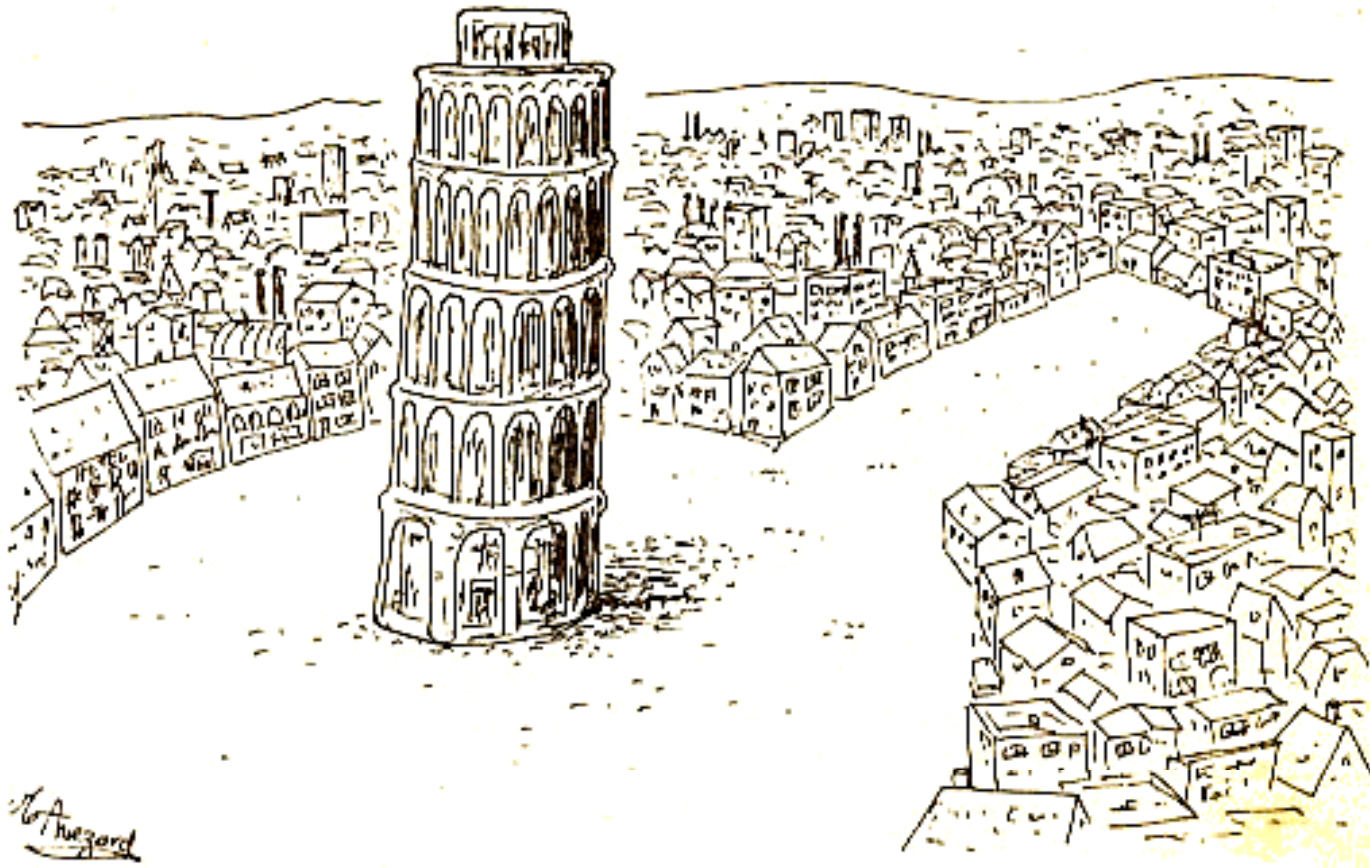


# Credit Portfolio Management

## Beispiel 3: Cash Account and Liabilities



# Lessons from Pisa



Sans parois.

© ZEVAR 1976 (LE PELERIN)