

Computational Tools for the Analysis of Market Risk



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Introduction



■ Who measures **Risk**?

- Financial institutions.
- Supervisory agencies.
- Rating agencies.

■ Why measure **Risk**?

- Control / manage risk associated with extreme events in the markets.
- Capital adequacy Regulations.

Introduction



■ In theory: Markets are efficient

- Absence of arbitrage opportunities.
- No systematic trends.
- Very short term memory.

■ Model: Black-Scholes

- Log of daily returns of an asset are distributed according to a normal distribution.
- Two parameters:
 - Risk free **interest rate**.
 - **Volatility**.

Financial Risks



■ Which are the sources of **Risk**?

- Market risks.
- Credit risks.
- Liquidity risks.
- Operational risks.
- Legal risks.

Measuring Risk



■ What (precisely) do we mean by **Risk**?

Uncertainty associated with unexpected fluctuations in the value of assets or liabilities of interest due to movements in financial markets.

■ How do we measure **Risk**?

- Value-at-Risk.
- Shortfall.
- MaxVaR.
- Unconditional / Conditional Measures

MatRisk



- **Matrisk** is an interactive tool to carry out Risk analysis
 - **Normal VaR.**
 - Parametric fit to **mixtures of normals.**
 - **Shortfall** analysis.
 - **MaxVaR** analysis.
- Entirely developed in MATLAB.
- Development in Java under way.

Value at Risk

- **Definition:** Estimate of worst loss in the value of a portfolio over a time horizon with a given probability.
 - Time horizon (holding period): Usually short (eg. 1 day)
 - Probability: Usually high (eg. 99 %).
- VaR = Percentile of returns distribution at a given probability level .

Sample-derived percentiles.



■ Discreteness of extreme events.

- No out-of-sample predictions.
- Influence of sample size.
- Sensitivity to time window.
- Highly variable estimations of VaR.

■ These disadvantages can be (partially) alleviated by a parametric approach.

Parametric Fit

- **Likelihood function:**

$$L(x_1, x_2, \dots, x_N) = \prod_{n=1}^N P(x_n)$$

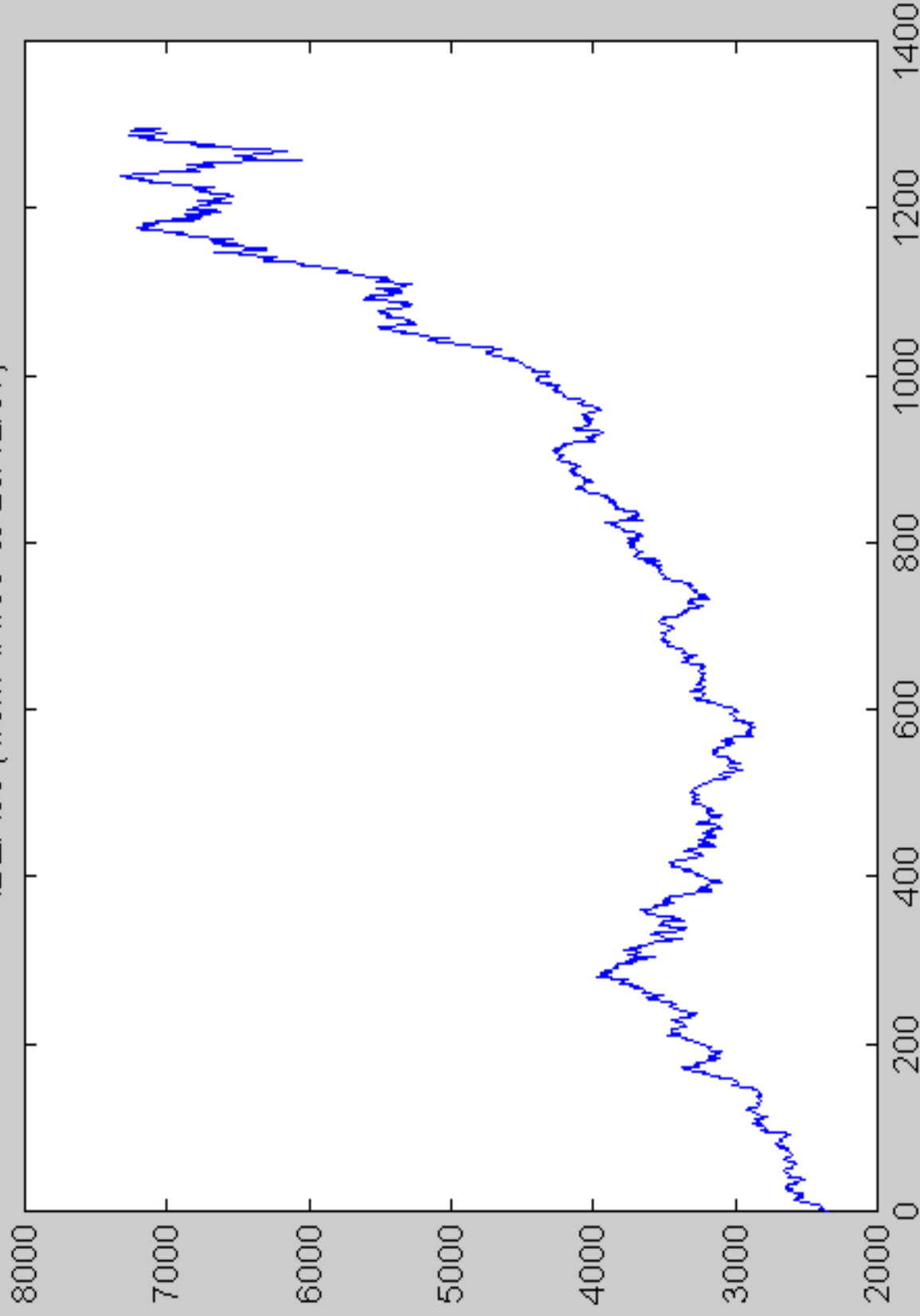
- **Fit:** Maximization of Log-likelihood.
- **Hypothesis for the pdf:** Normal, mixture of normals, hyperbolic.

Figure No. 1

File Edit Tools Window Help

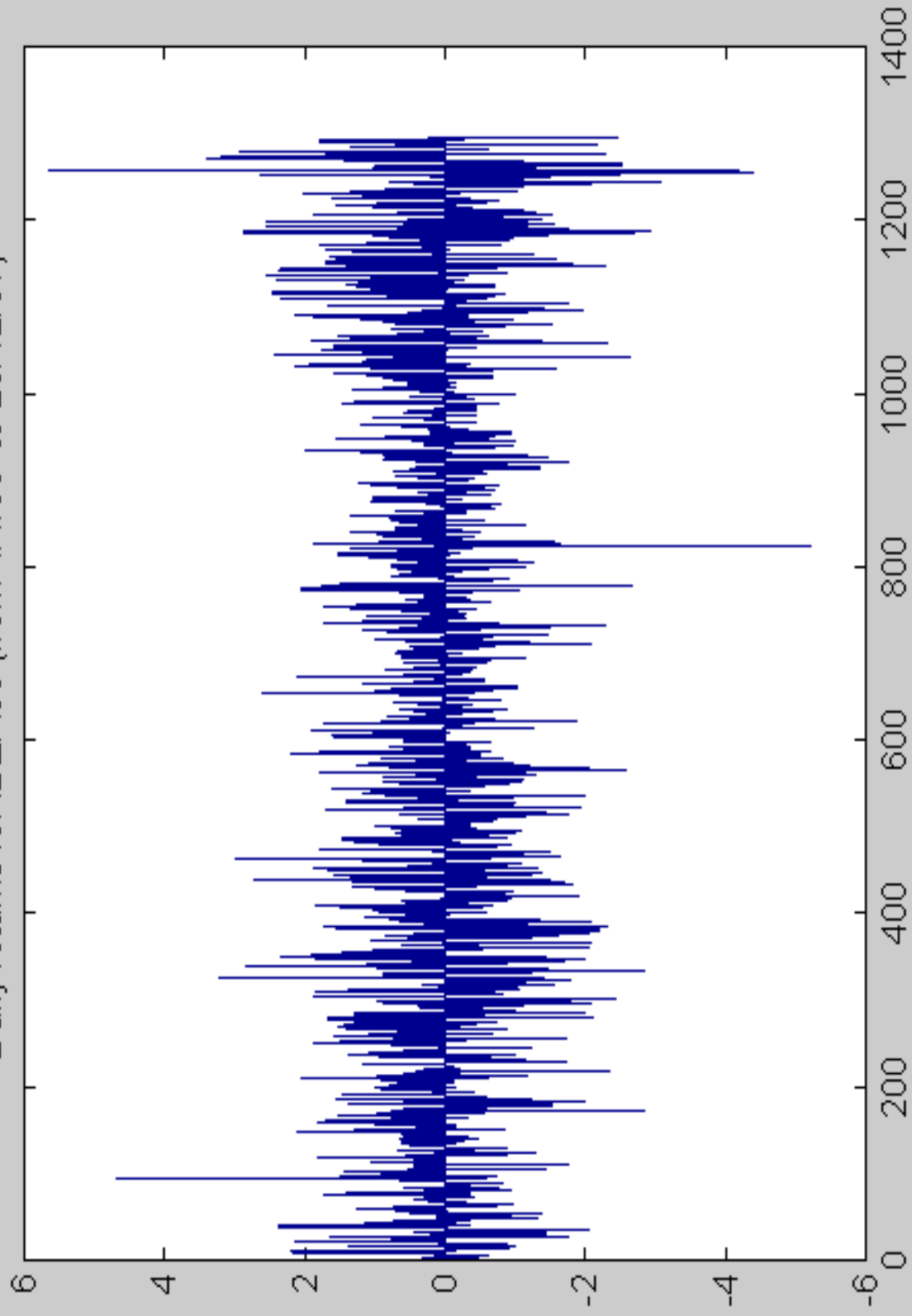


IBEX35 (from 4/1/93 to 23/12/97)





Daily returns for IBEX35 (from 4/1/93 to 23/12/97)



Normal model



■ Probability distribution

$$N(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

■ Problems:

- | Heavy tails
- | Non-independence of returns

Figure No. 1: VaR: Mixture of Gaussians

File Edit Tools Window Help

Data file

ibex35.txt

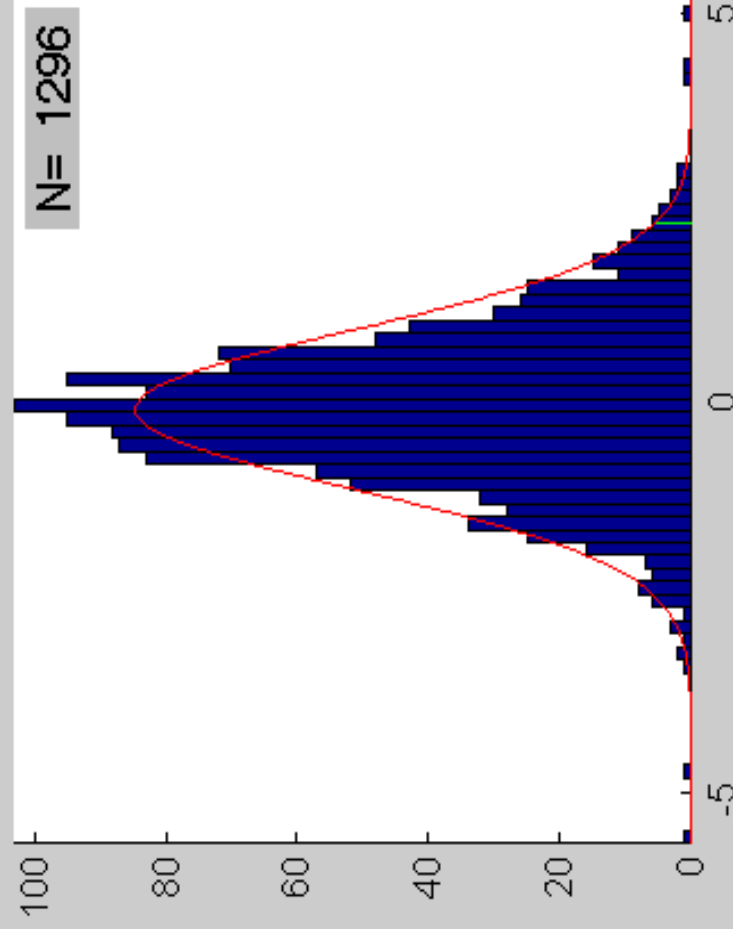
Number of Gaussians

1

Fit

KS statistic = 0.12

N= 1296



90

100

VaR (99%) = 2.29421
Sample VaR = 2.48554
 $E[x / x > 2.29421] = 2.64153$

QQPlot

$p = 1$
 $\mu = -0.0901416$
 $\sigma = 1.02493$

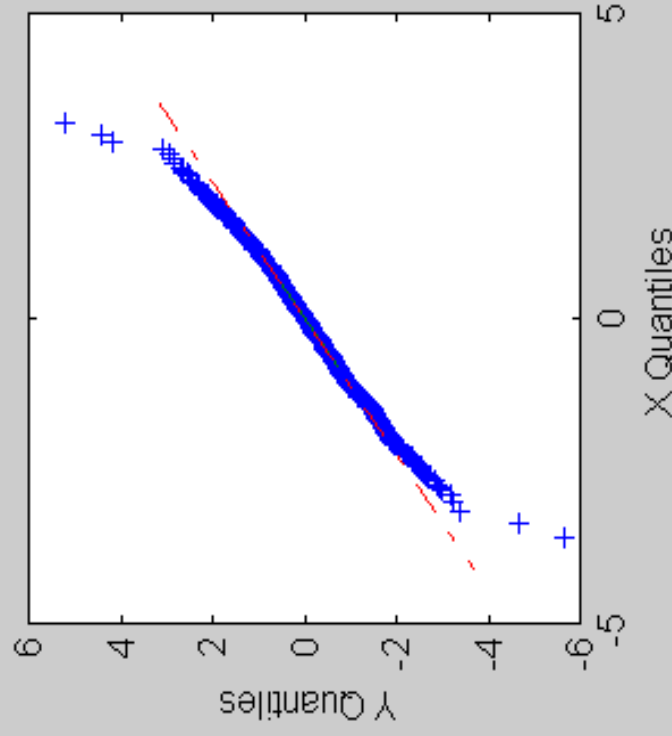
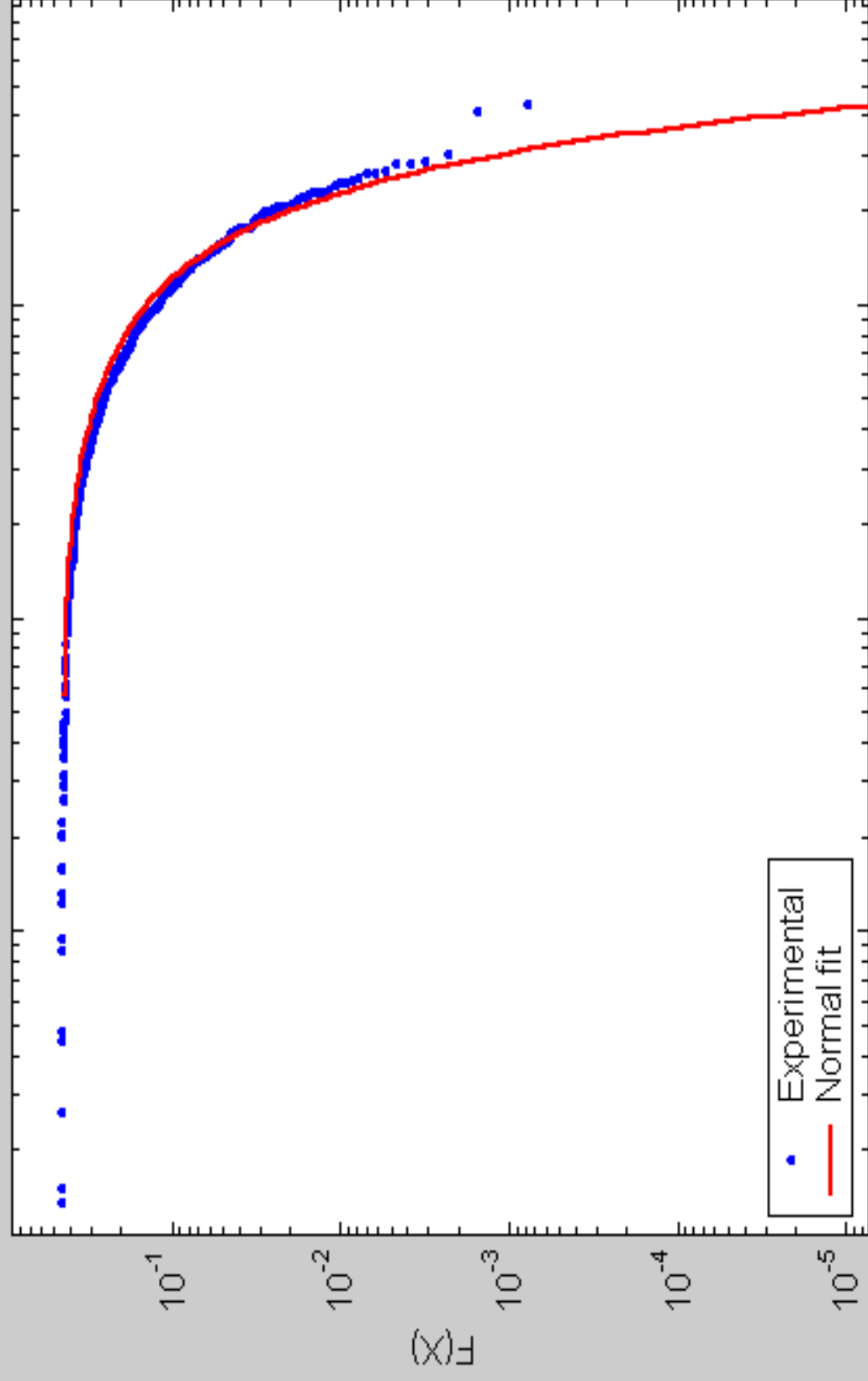


Figure No. 2

File Edit Tools Window Help



Fit to Normal distribution



Mixture of Gaussians

■ Probability distribution

$$P(x) = \sum_{k=1}^K p_k \mathbf{N}(x; \mu_k, \sigma_k)$$

■ **Warning:** The log-likelihood function does not have a global minimum for mixtures of Gaussians.

■ **Solution:** Use a modified log-likelihood function.
(Hamilton 1991)

Mixture model

- X_T is taken from a mixture of normals

$$X_T = \begin{cases} N(\mu_1, \sigma_1) & \text{with probability } p \\ N(\mu_2, \sigma_2) & \text{with probability } (1 - p) \end{cases}$$

- Intuitively appealing: Close to BS world
- Useful to capture extreme events (Ej. In VaR)
- **But:**
 - No clear dynamical picture (a jump process?)
 - More parameters to fit (Overfitting?)
 - Tails are still exponential

Figure No. 1: VaR: Mixture of Gaussians

File Edit Tools Window Help

Data file

ibex35.txt

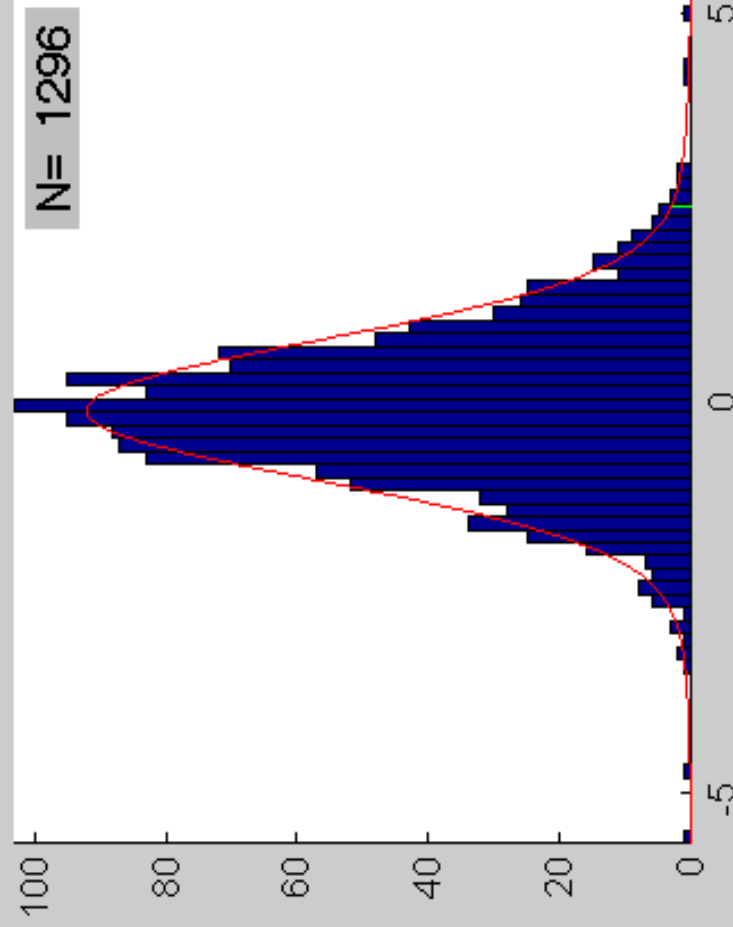
Number of Gaussians

2

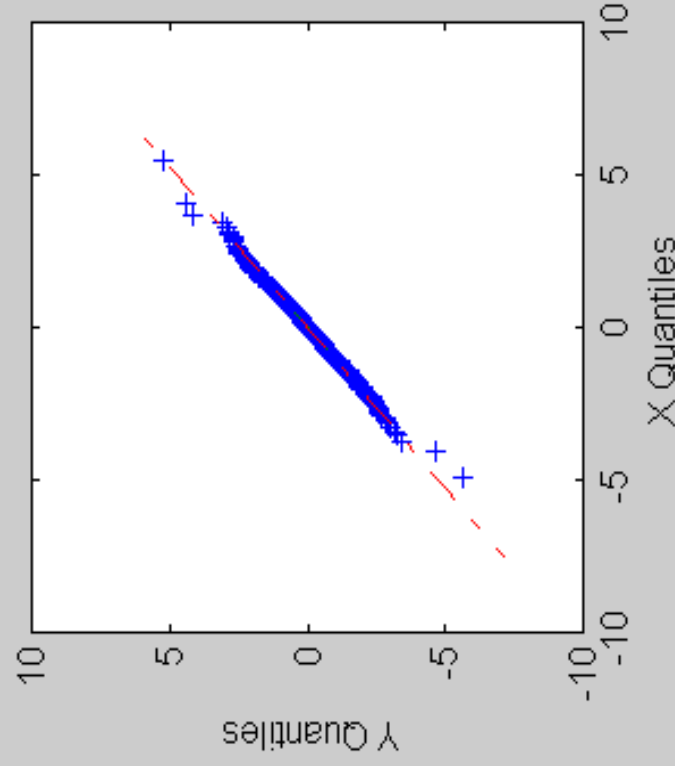
Fit

KS statistic = 0.92

N= 1296



QQPlot



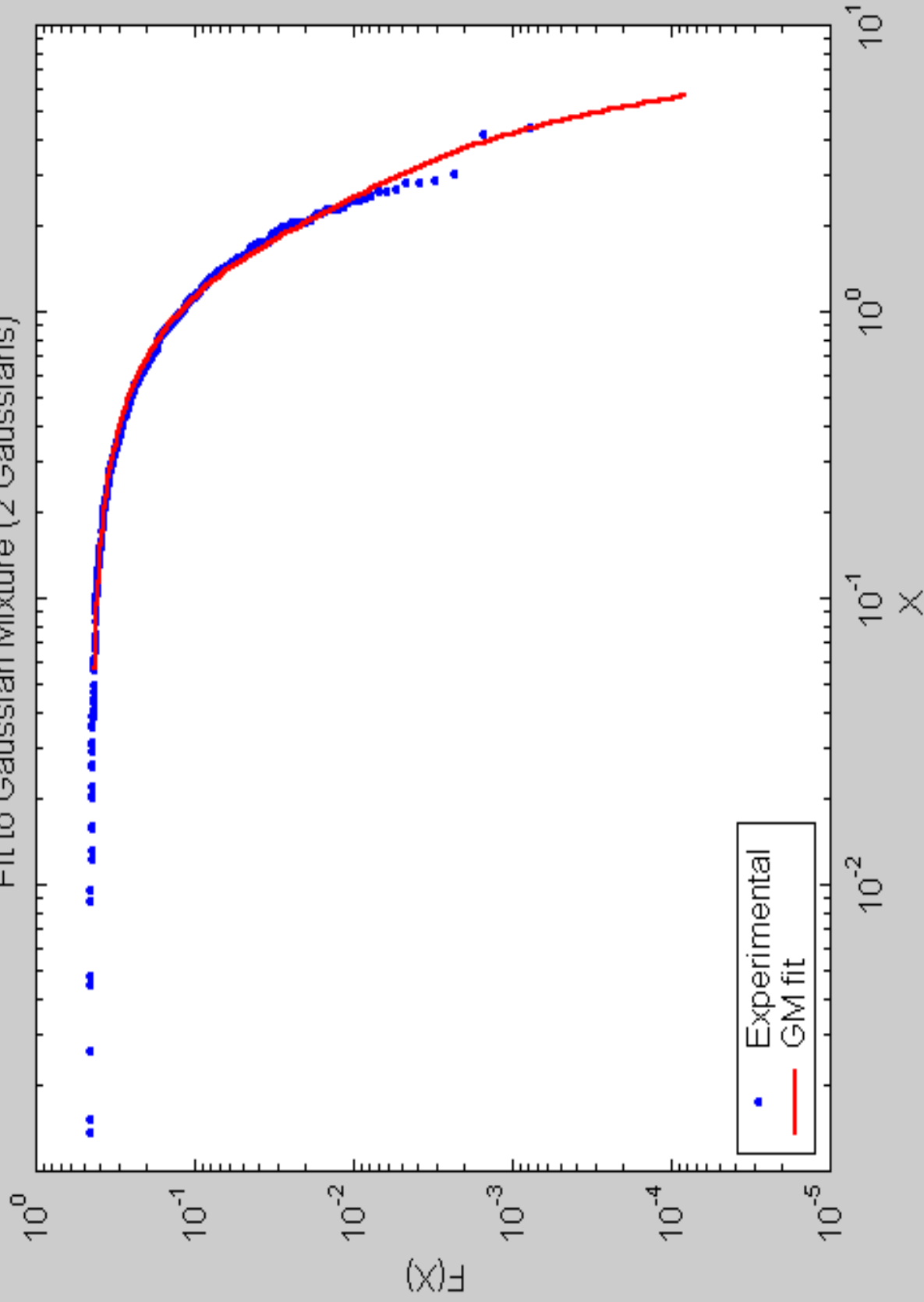
VaR (99%) = 2.52694
Sample VaR = 2.48554
 $E[x / x > 2.52694] = 3.27123$

Figure No. 2

File Edit Tools Window Help



Fit to Gaussian Mixture (2 Gaussians)



Data file

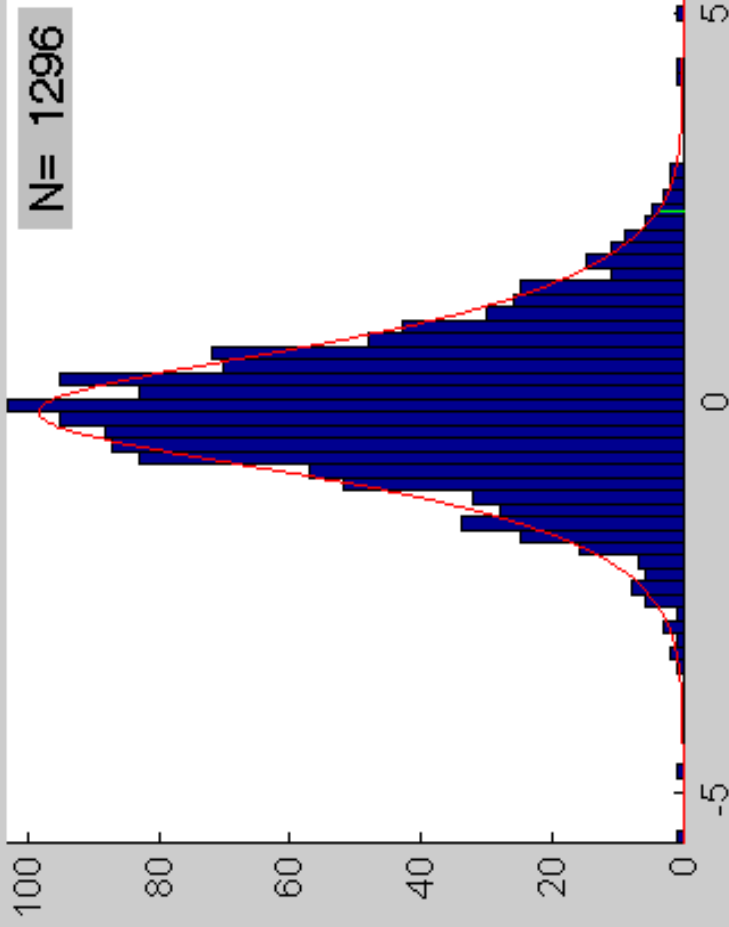
ibex35.txt

Number of Gaussians

3

Fit

KS statistic = 0.99



90

VaR (99%) = 2.46005
 Sample VaR = 2.48554
 $E[x / x > 2.46005] = 3.20786$

100

$\rho = 0.254784$ 0.0196478 0.7255568
 $\mu = -0.140678$ 0.0895231 -0.0772471
 $\sigma = 0.568491$ 2.63393 1.06877

QQPlot

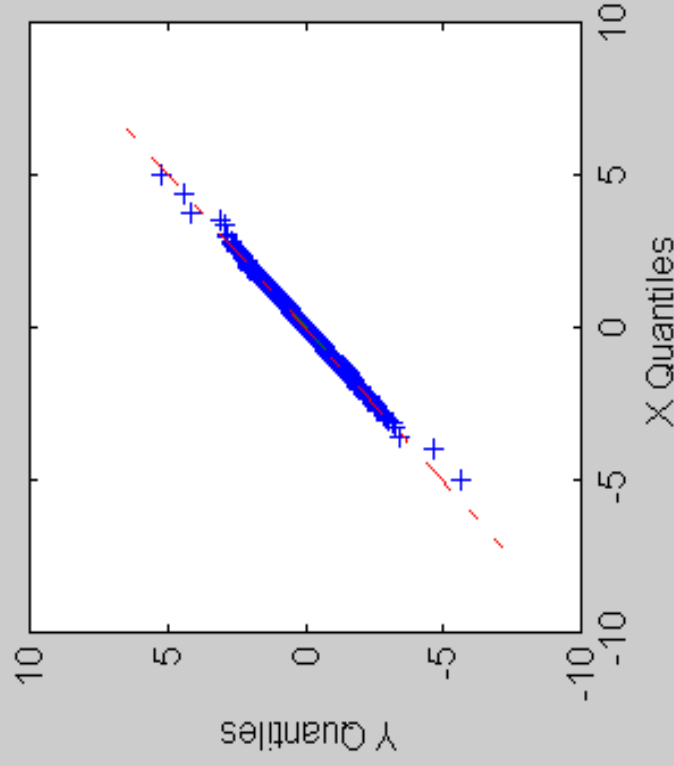
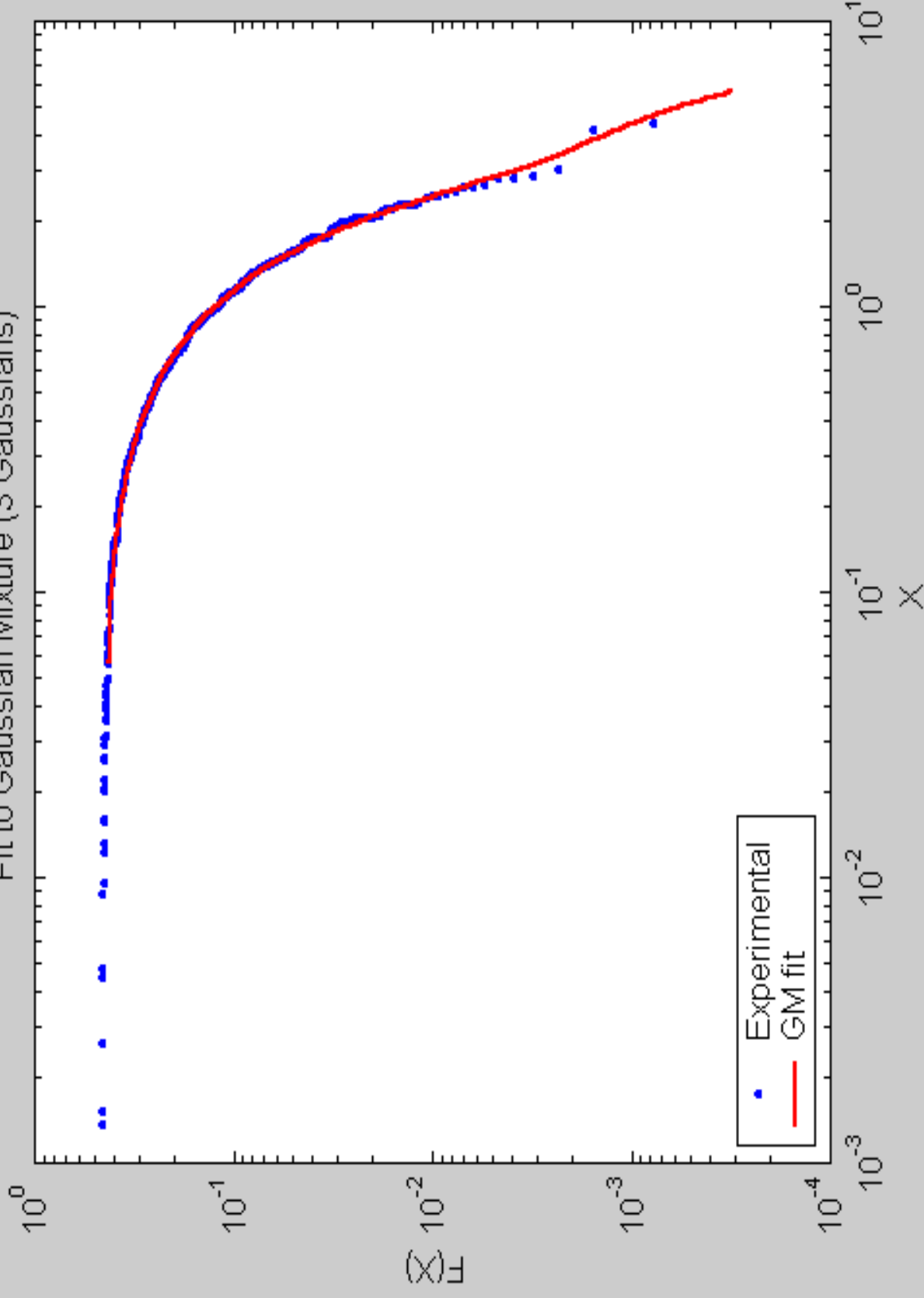


Figure No. 3

File Edit Tools Window Help



Fit to Gaussian Mixture (3 Gaussians)



Hyperbolic Distribution

■ Probability distribution

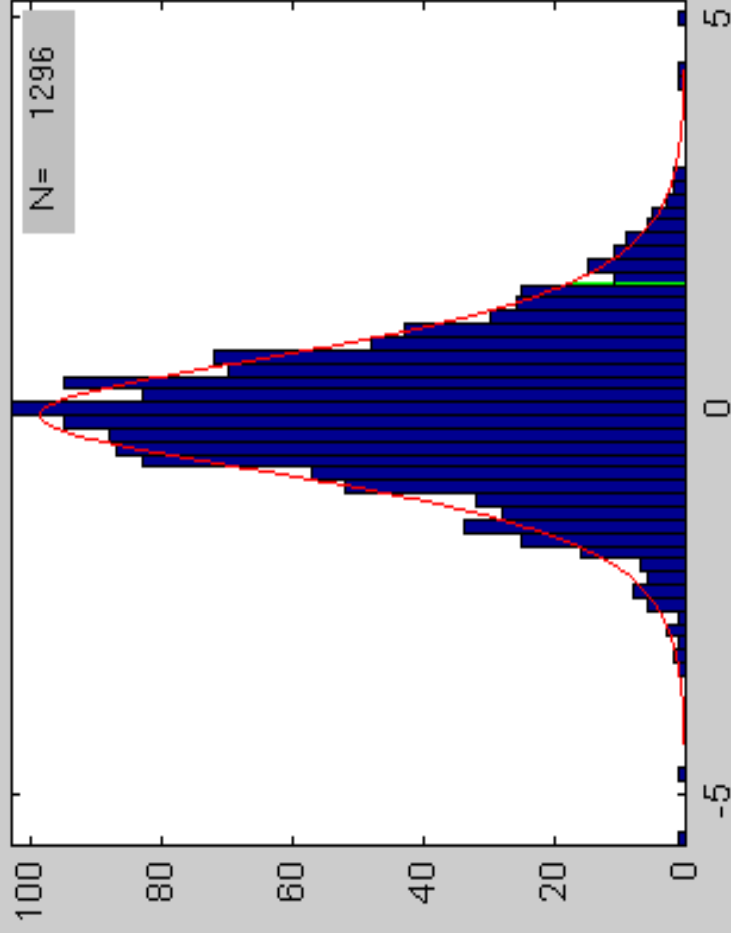
$$P(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1(\sqrt{\alpha^2 - \beta^2})} \exp\left(-\alpha\sqrt{\delta^2 + (x - \mu)^2} + \beta(x - \mu)\right)$$

- **Limiting cases:** Normal, Laplace, exponential, generalized inverse Gaussian.
- It is the stationary distribution of a continuous Markov process.

Data file

ibex35.txt

Fit



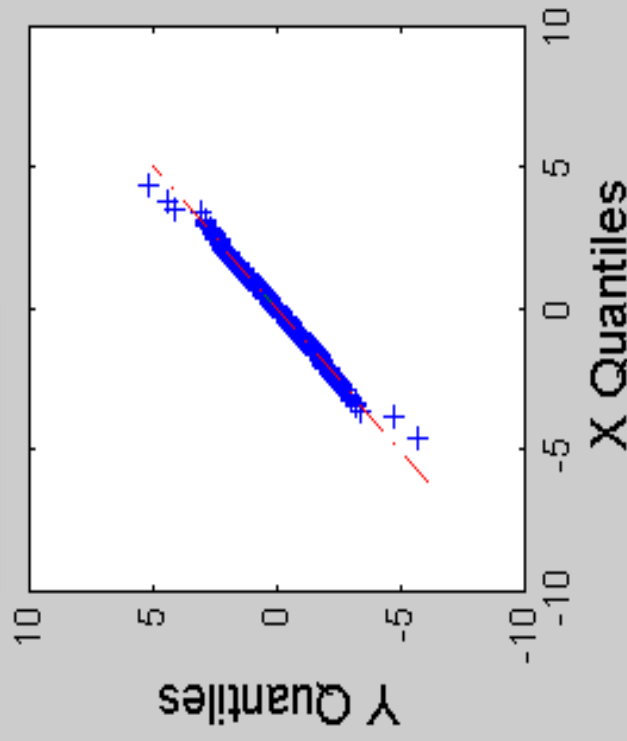
90 [] 100

VaR (95%) = 1.59336
 Sample VaR = 1.57846
 E[x / x > 1.59336] = 2.19357

KS statistic 0.99

$\alpha = 1.84159$
 $\beta = 0.0651013$
 $\delta = 1.02963$
 $\mu = -0.157995$

QQPlot



VaR: Advantages / drawbacks



■ Advantages:

- Summarizes exposure to market risk in a single number.
- Simple, intuitive, easy to understand.

■ Drawbacks:

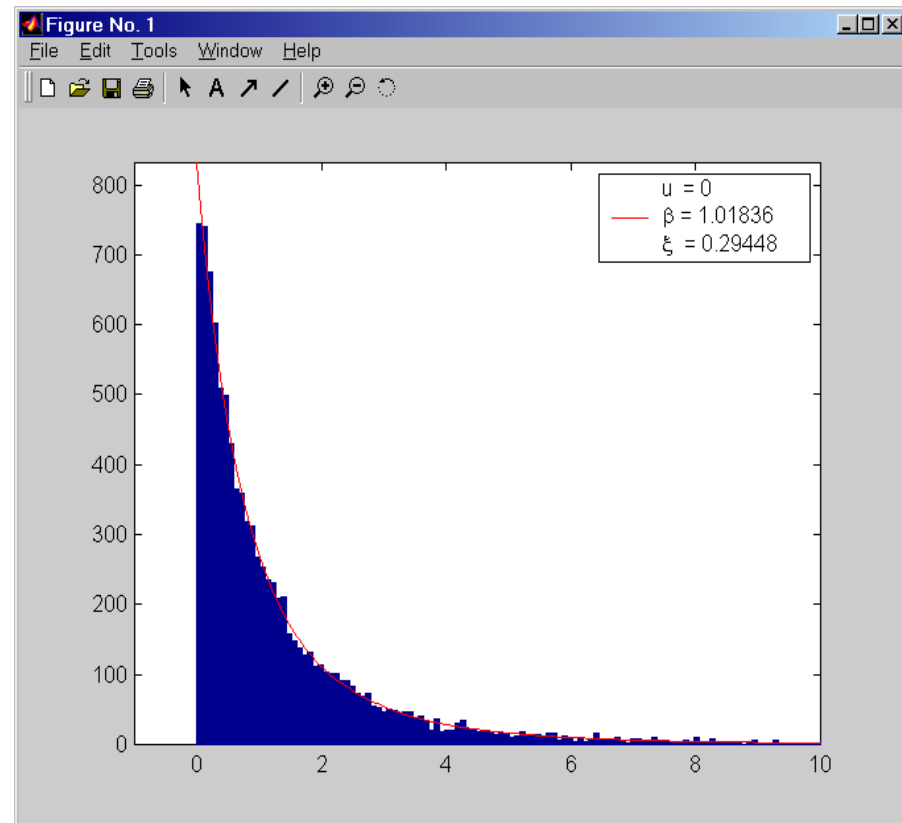
- It is not subadditive for heavy-tailed distributions.
- Problems to capture events in the tails.
- Problems with time aggregation.

Beyond VaR: Shortfall

■ Generalized Pareto Distribution

$$P(x) = \frac{1}{\beta} \left(1 + \frac{\xi}{\beta} (x - u)_+ \right)^{-\left(1 + \frac{1}{\xi}\right)}$$

Objective: Fit the **tails** of the distribution, assuming algebraic decay.



Estimating Shortfall

■ **Definition:** *Shortfall* $\equiv E[x / x > u]$

■ **Mean excess**
$$e(u) = \frac{1}{N_u} \sum_{n=1}^{N_u} (x_n - u)_+$$

- Rank-ordered points: $x_1 \geq x_2 \geq \dots \geq x_N$
- Choose u so that mean excess is linear in u .
- Parametric fit to GPD by log-likelihood maximization.
- Find shortfall from GPD.

Figure No. 1: Shortfall: Generalized Pareto

File Edit Tools Window Help

Data file

ibex35.txt

Fit

Threshold for excess

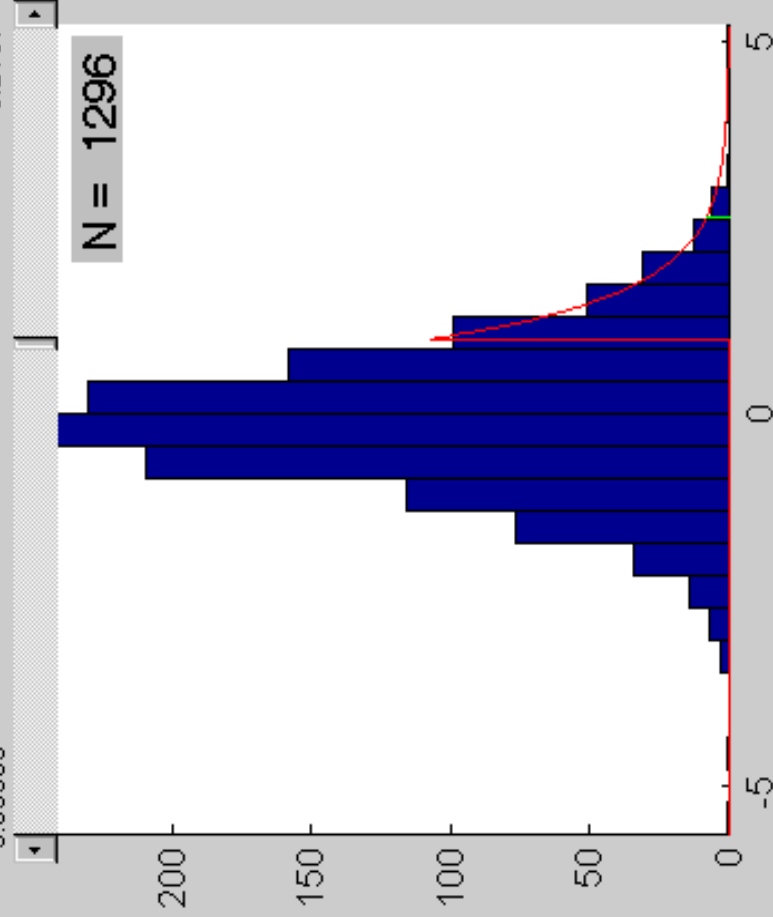
1

KS statistic = 0.78

5.2157

-5.65565

N = 1296



87.8

100

VaR (99.00%) = 2.62866
Sample VaR = 2.48554
 $E[x / x > 2.62866] = 3.30093$

Mean Excess Plot

$u = 1$
 $\beta = 0.63975$
 $\xi = 0.0141361$

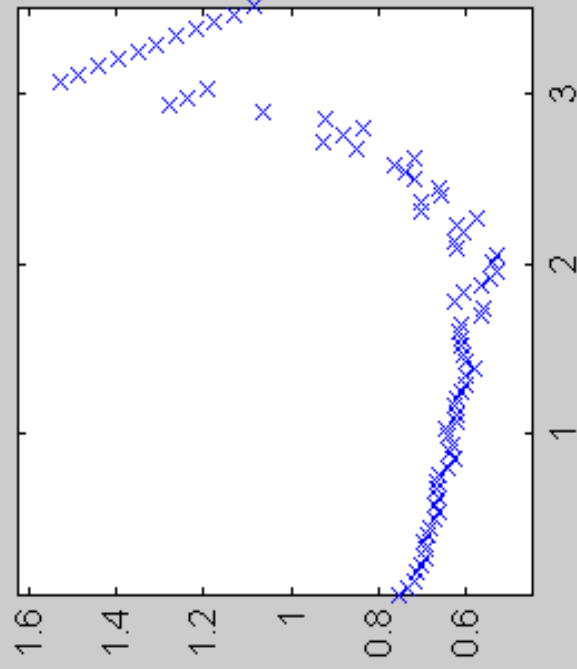


Figure No. 3

File Edit Tools Window Help



Fit to Generalized Pareto distribution ($u=1$)

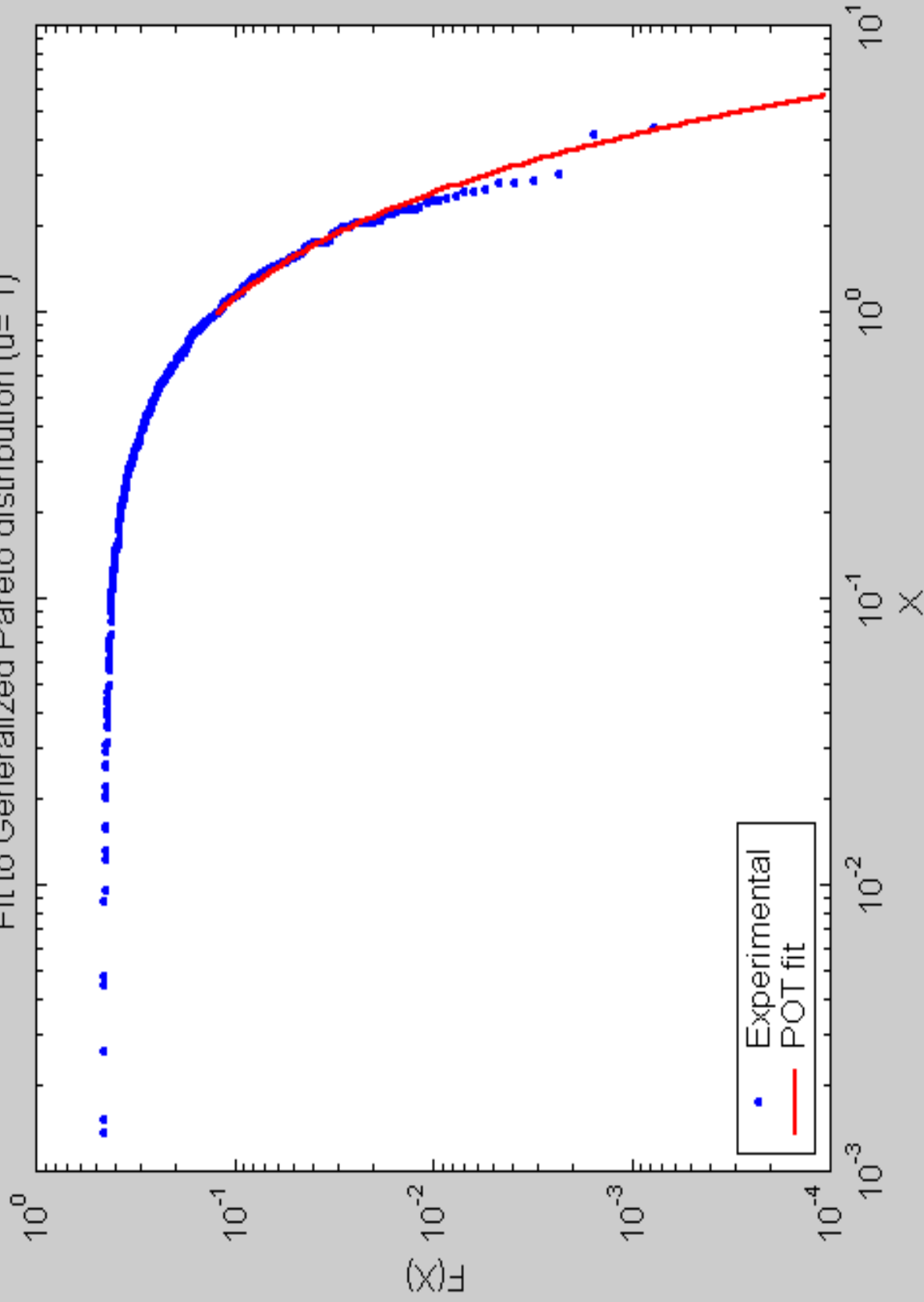


Figure No. 1: Shortfall: Generalized Pareto

File Edit Tools Window Help

Data file

ibex35.txt

Fit

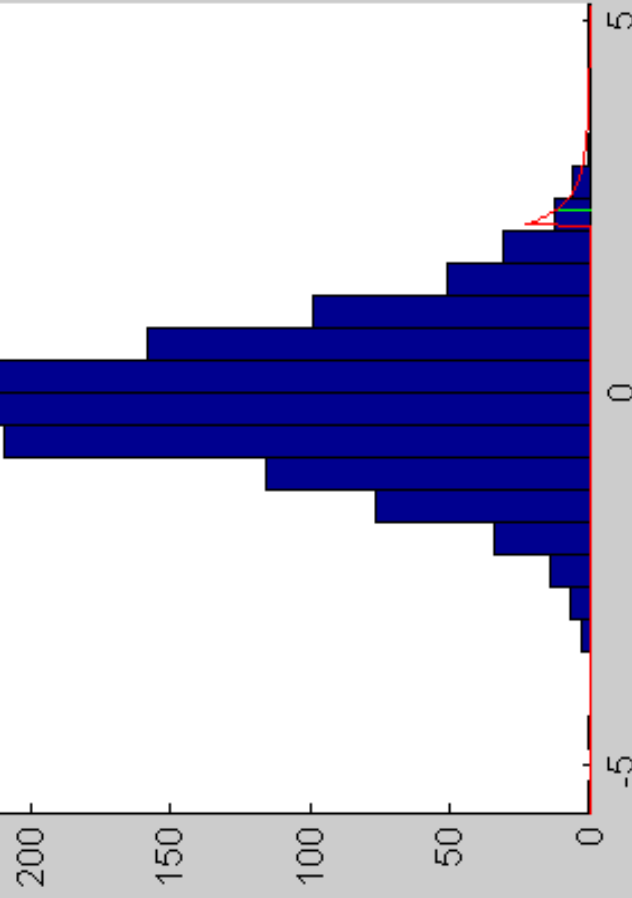
Threshold for excess

2.25

-5.65565

5.2157

N = 1296



98.3

100

VaR (99.00%) = 2.45335
Sample VaR = 2.48554
 $E[X/x > 2.45335] = 3.20551$

KS statistic = 0.96

$u = 2.25$
 $\beta = 0.38298$
 $\xi = 0.386374$

Mean Excess Plot

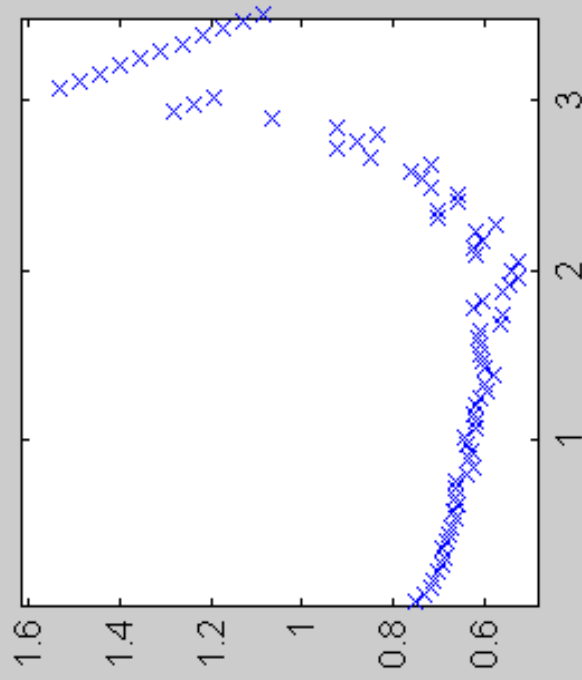
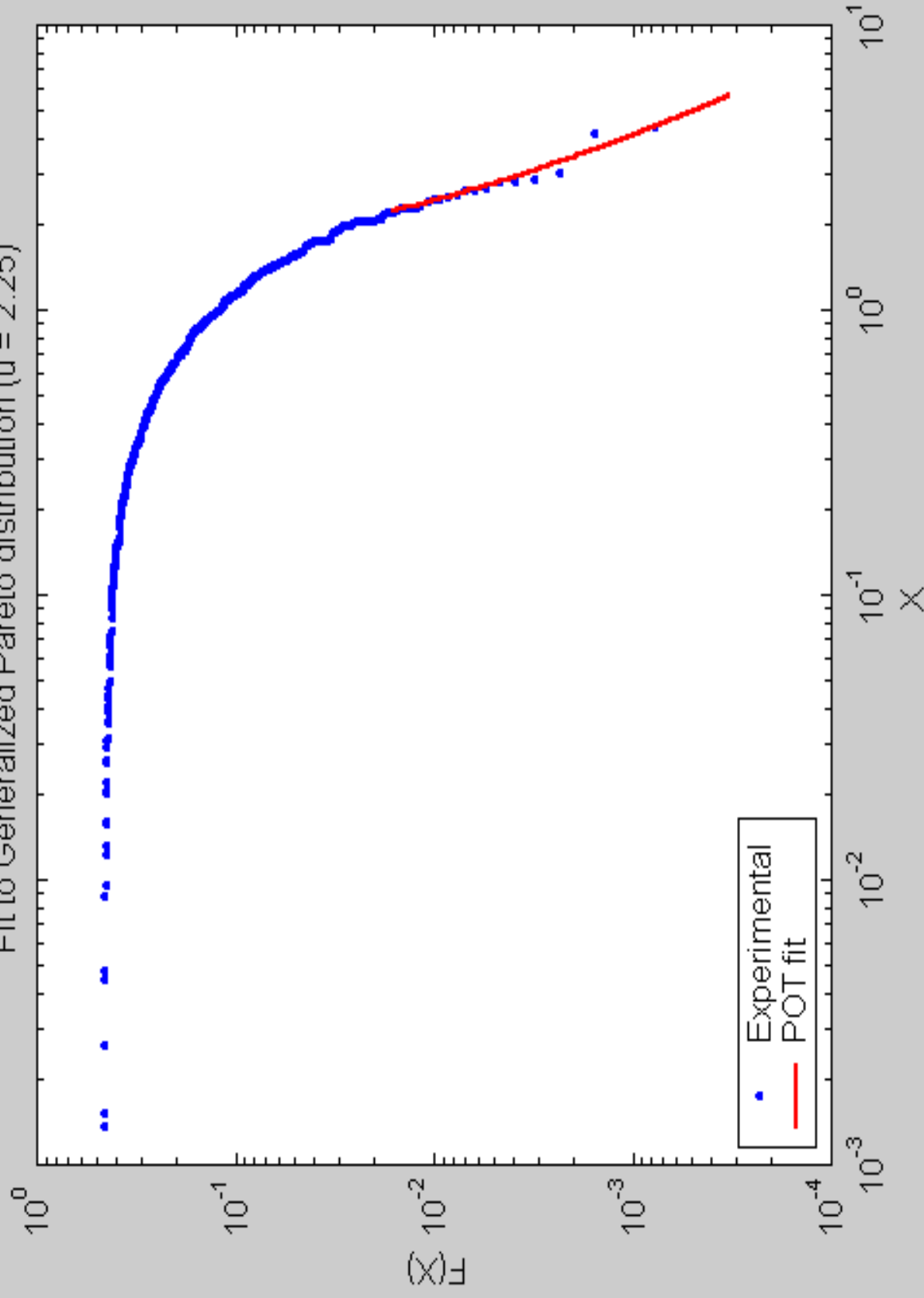


Figure No. 3

File Edit Tools Window Help



Fit to Generalized Pareto distribution ($u = 2.25$)



Shortfall in finance

- The value of ξ in finance is typically in the interval $[0.2, 0.7]$
- **Tail index:** $\alpha = 1/\xi$; $1.5 < \alpha < 5$
- **VaR / ShortFall**

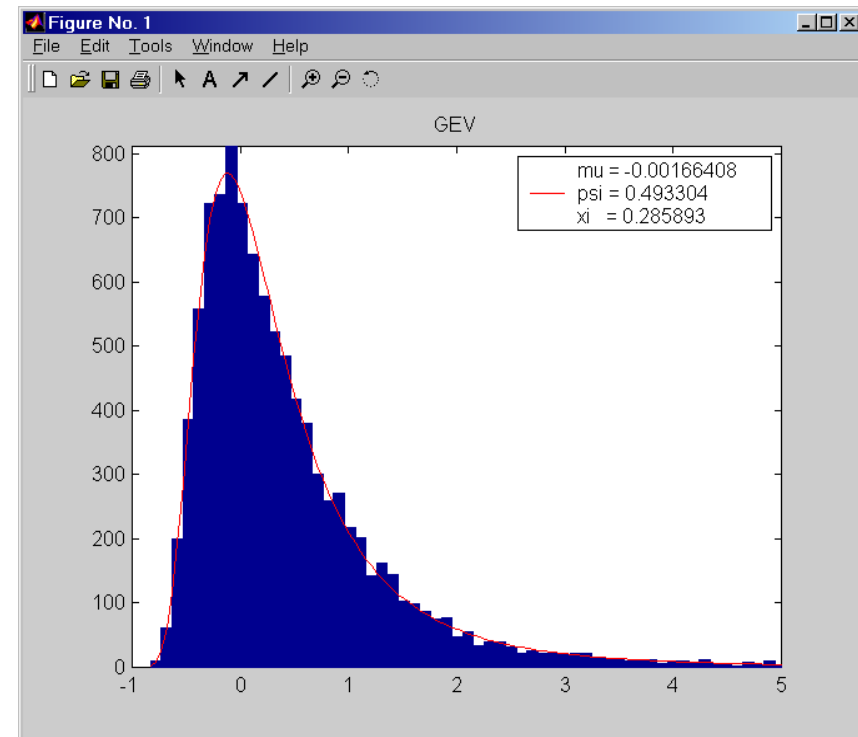
$$E[x / x > VaR] \approx \frac{\alpha}{\alpha - 1} VaR$$

Beyond VaR: MaxVar

Generalized Extreme Value Distribution

$$P(x) = \frac{1}{\psi} \left(1 + \frac{\xi}{\psi} (x - \mu) \right)^{-\left(1 + \frac{1}{\xi}\right)} e^{-\left(1 + \frac{\xi}{\psi} (x - \mu)\right)^{\frac{1}{\xi}}}$$

Objective: Fit **maxima** of the distribution, assuming **independent events**.

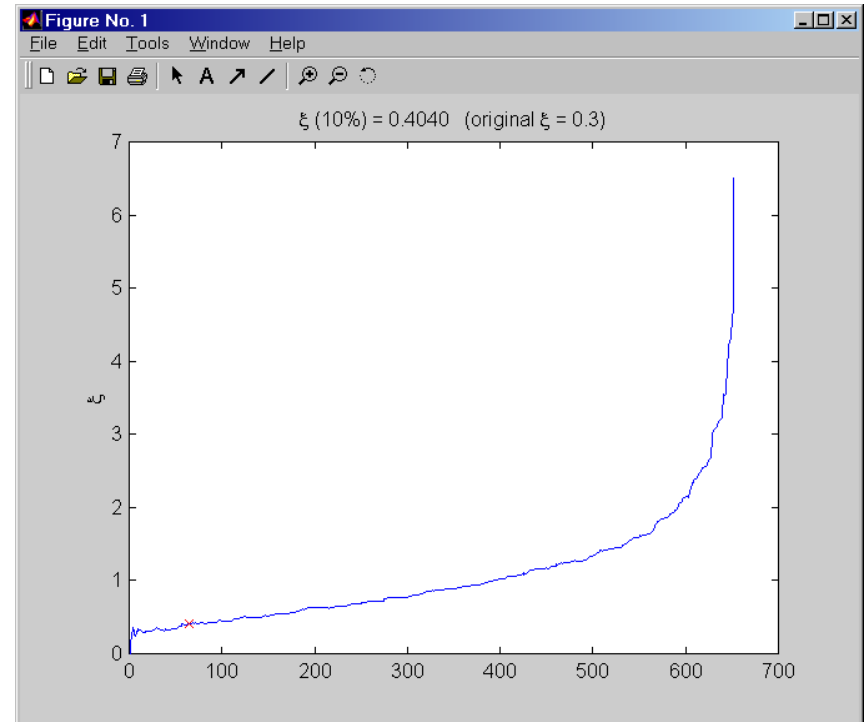


Estimate of shape parameter

■ Hill Estimator for ξ

- Rank-ordered points: $x_1 \geq x_2 \geq \dots \geq x_N$
- $k \gg 1$; $k \ll N$

$$\xi_{Hill} \equiv \frac{1}{k} \sum_{j=1}^k \log \frac{x_j}{x_k}$$



Data file

ibex35.txt

Set initial parameter values

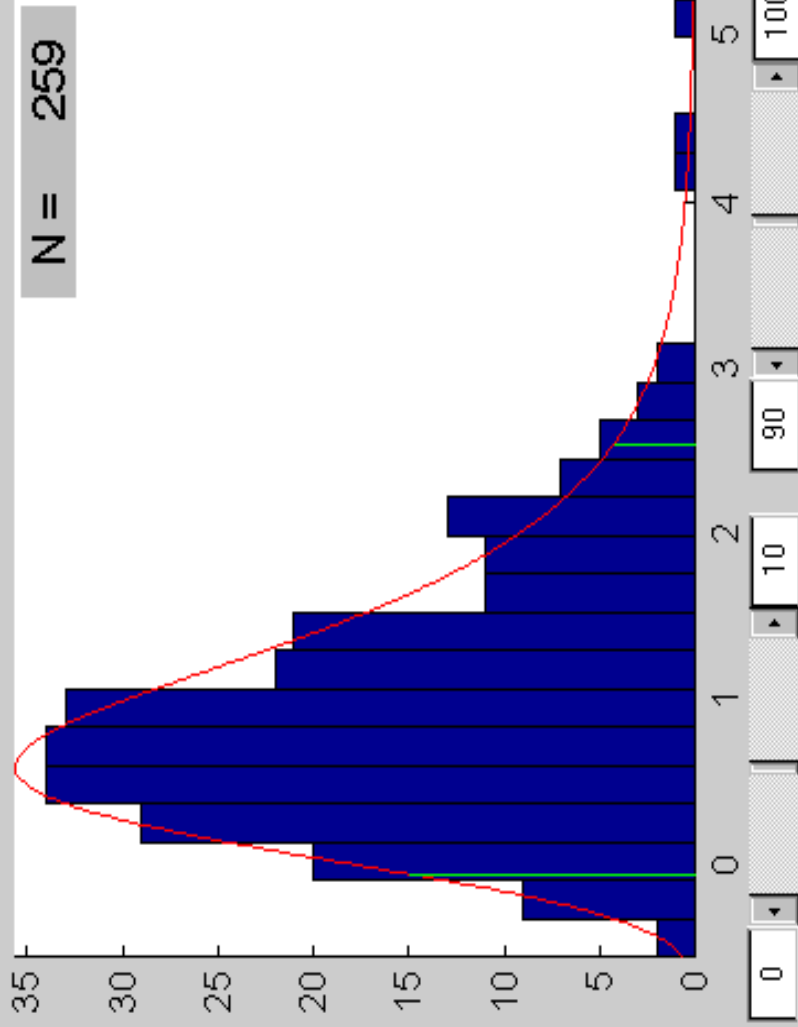
Fit

Size of groups

5

1 51

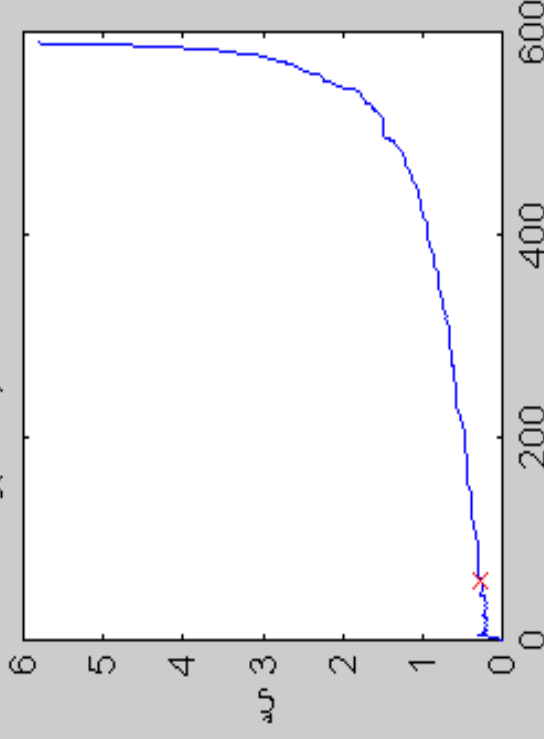
KS statistic = 0.99



$\mu = 0.608381$
 $\psi = 0.61788$
 $\xi = 0.0324053$

Hill Plot

$\xi(10\%) = 0.280176$



Percentls. [5.00%,95.00%] = [-0.0576397,2.53483]
 Sample Percentiles = [-0.0628672,2.4714]
 $E[X | -0.0576397 < x < 2.53483] = 0.926376$

Data file

ibex35.txt

Set initial parameter values

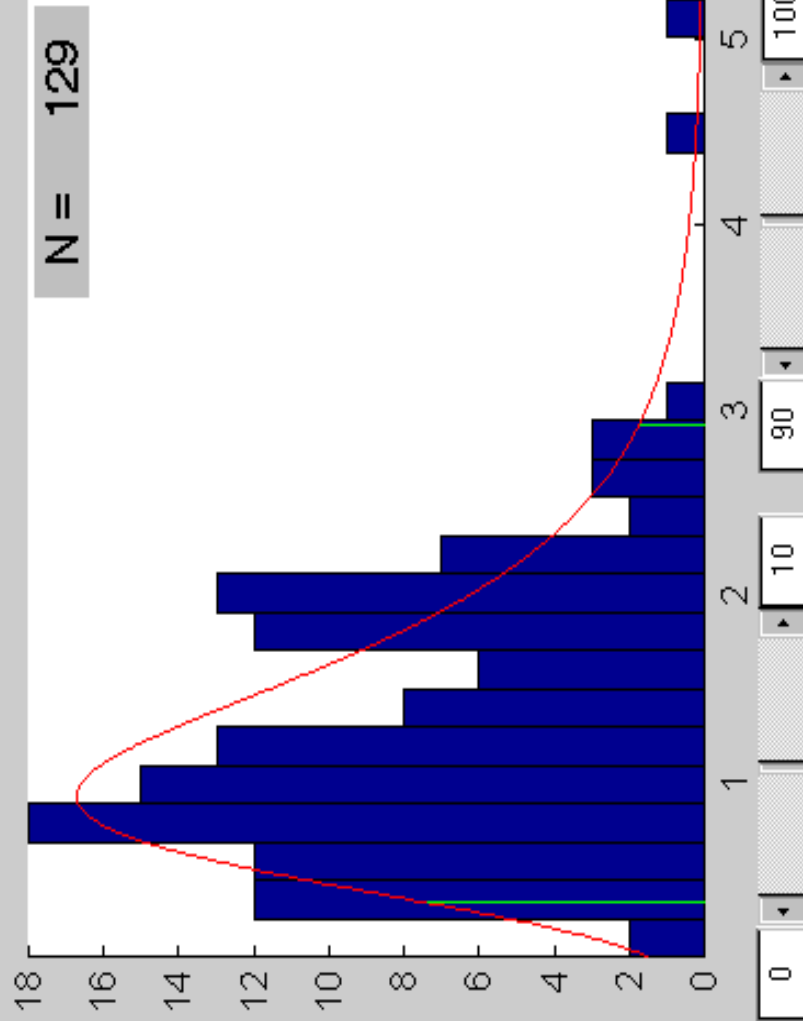
Fit

Size of groups

10

1 51

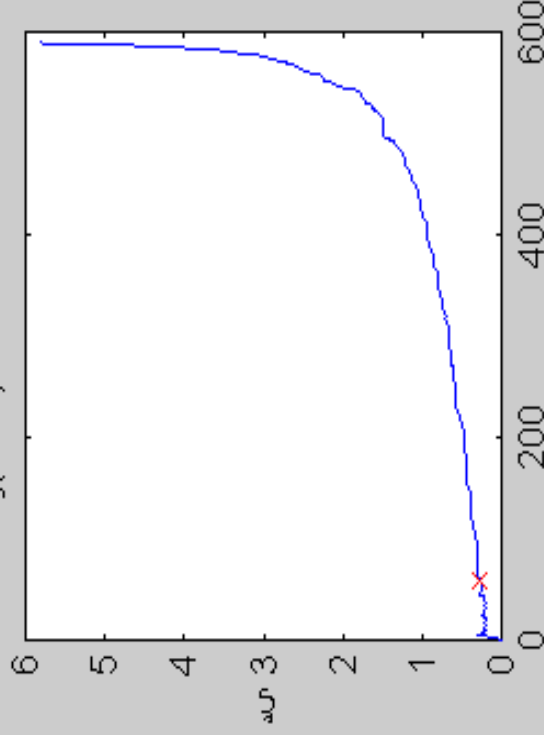
KS statistic = 0.47



$\mu = 0.9666631$
 $\psi = 0.586434$
 $\xi = 0.0773592$

Hill Plot

$\xi(10\%) = 0.280176$



Percentls. [5.00%,95.00%] = [0.349752,2.92481]
 Sample Percentiles = [0.360391,2.65263]
 $E[x / 0.349752 < x < 2.92481] = 1.28463$

MaxVar: A balance



- Based in **universal properties** of distributions of maxima of iid random variables.
- Accounts for **extreme events** (beyond sample).
- Does **not assume** an **a priori** functional **form** for the original probability distribution.
- Despite the fact that the parameters of the GEV distribution vary greatly, the **percentiles** measured **are** fairly **stable**.

Summary IBEX35 (I)

- Comparison between different measures of VaR

	KS Statistic	95%	99%	99.9%	99.99%
Sample VaR		1.58	2.49	4.40	5.22
GM1	0.12	1.60	2.29	3.36	3.72
GM2	0.92	1.54	2.53	4.25	5.63
GM3	0.99	1.58	2.46	4.41	6.86
HYP	0.99	1.59	2.57	3.91	5.30
GP (u=1)	0.78	1.57	2.63	4.18	5.78
GP (u=2.25)	0.96	***	2.45	4.16	8.34

Summary IBEX35 (II)

- Comparison between different measures of **Shortfall**

	KS Statistic	95%	99%	99.9%	99.99%
GM1	0.12	2.02	2.64	3.07	3.97
GM2	0.92	2.16	3.27	4.86	6.13
GM3	0.99	2.17	3.21	5.51	7.69
HYP	0.99	2.20	3.15	4.49	5.80
GP (u=1)	0.78	2.23	3.30	4.87	6.50
GP (u=2.25)	0.96	***	3.20	5.98	12.79

Real markets



■ In practice markets are

➤ Not efficient:

➤ Memory effects (short/long term?).

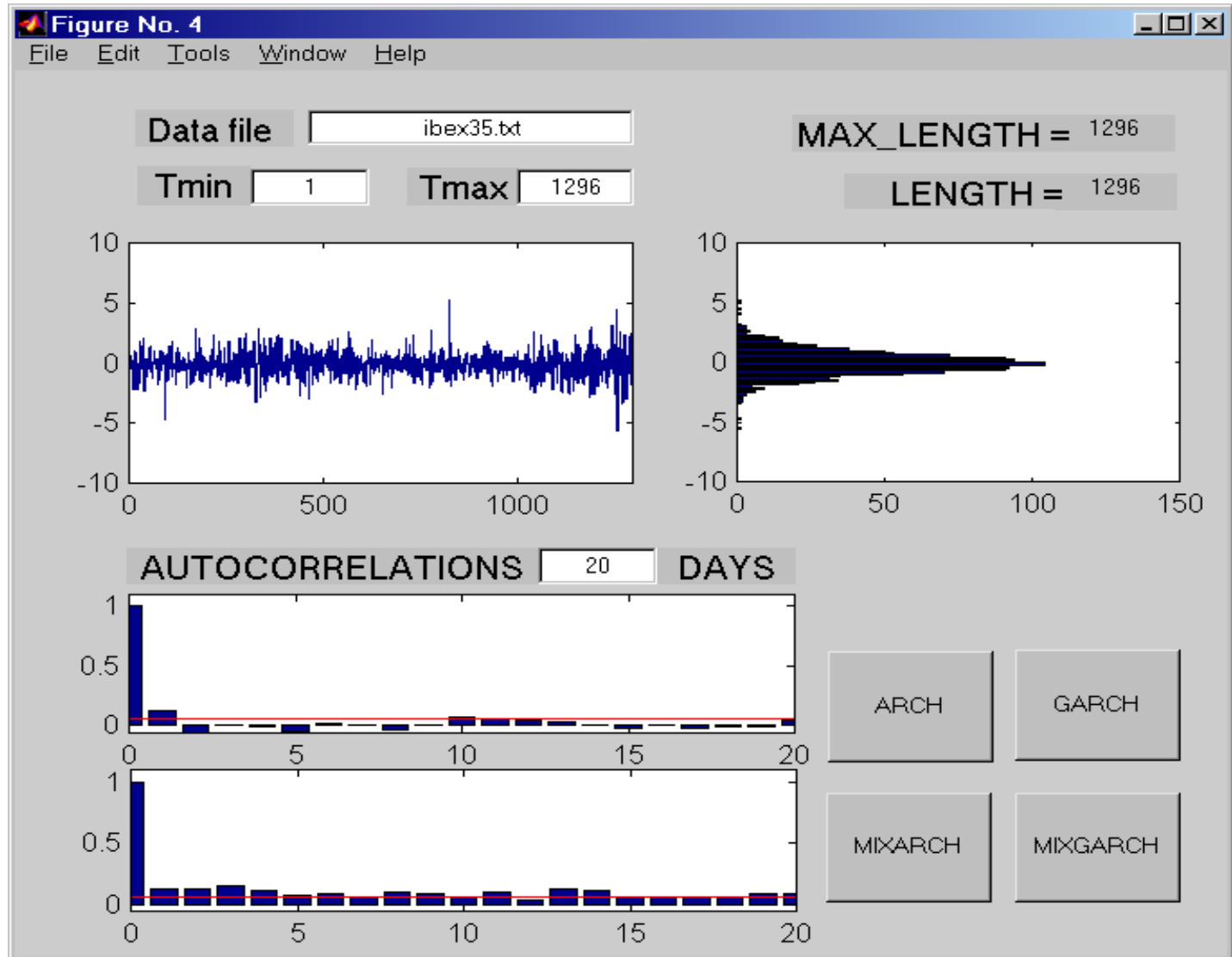
➤ Very unpredictable (at least sometimes)

➤ **Extreme events** are more frequent than what the normal models predict.

➤ Occurrence of **crashes**.

➤ **Abrupt changes** in economic paradigm.

Memory effects (IBEX 35)



Stationary random time series

- Consider the time series

$$X_0, X_1, X_2, \dots, X_{t-1}, X_t, \dots$$

- The series exhibits randomness.
- The process is covariance-stationary:

- Mean is time independent $E[X_t] = \mu$

- Autocovariance is independent of time-translations

$$E[(X_{t+\tau} - \mu)(X_t - \mu)] = \gamma_\tau$$

Conditional heteroskedasticity

- Model the time-structure of the volatility with an ARCH process

$$X_t = \phi^+ \cdot \mathbf{X}_t^{(m)} + u_t$$

$$u_t = \sigma_t Z_t$$

- The quantities Z_t are assumed to be iidrv's.
- The volatility follows an AR(q) process

$$\sigma_t^2 = \kappa + \alpha^+ \cdot [\mathbf{u}^2]_t^{[q]}$$

AR(1) / ARCH(1) for IBEX35

- The maximum-likelihood fit of the time-series IBEX35 yields the model

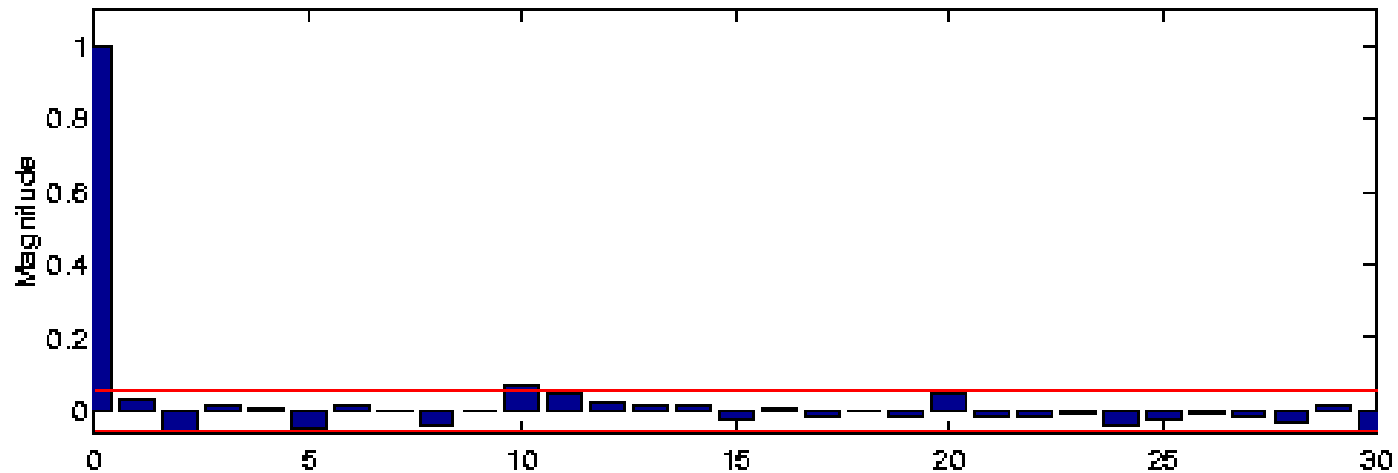
$$\hat{X}_t = 0.1129\hat{X}_{t-1} + \sigma_t Z_t$$

$$\sigma_t^2 = 0.9097 + 0.11118(\hat{X}_{t-1} - 0.1129\hat{X}_{t-2})^2$$

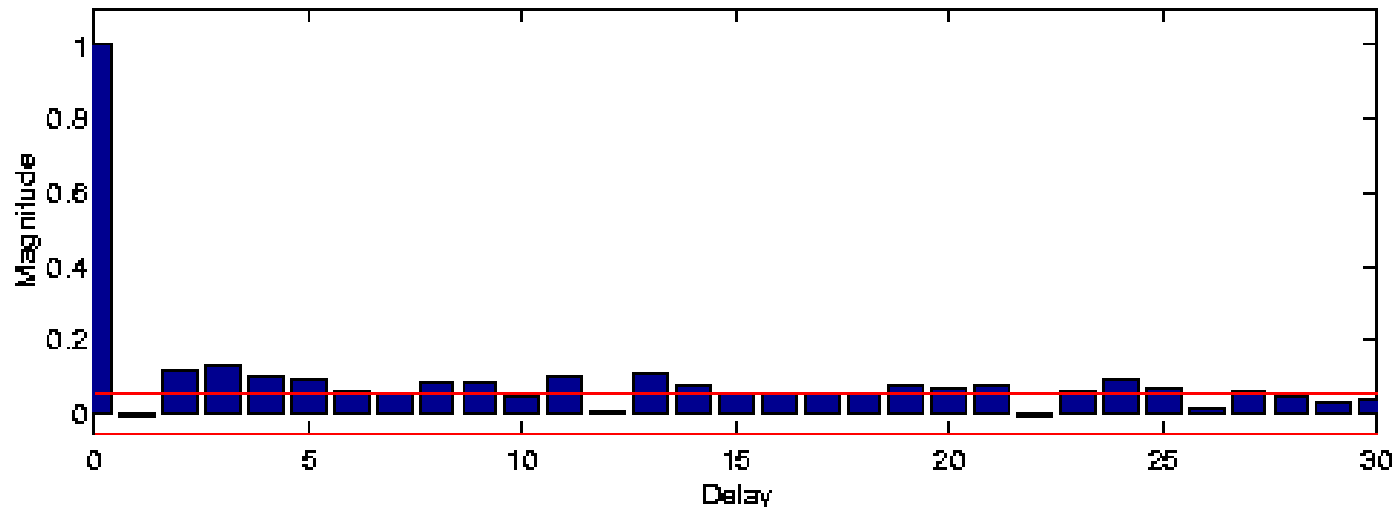
- The quantities Z_t are assumed to follow a $N(0,1)$ distribution.

Residual correlations: ARCH(1)

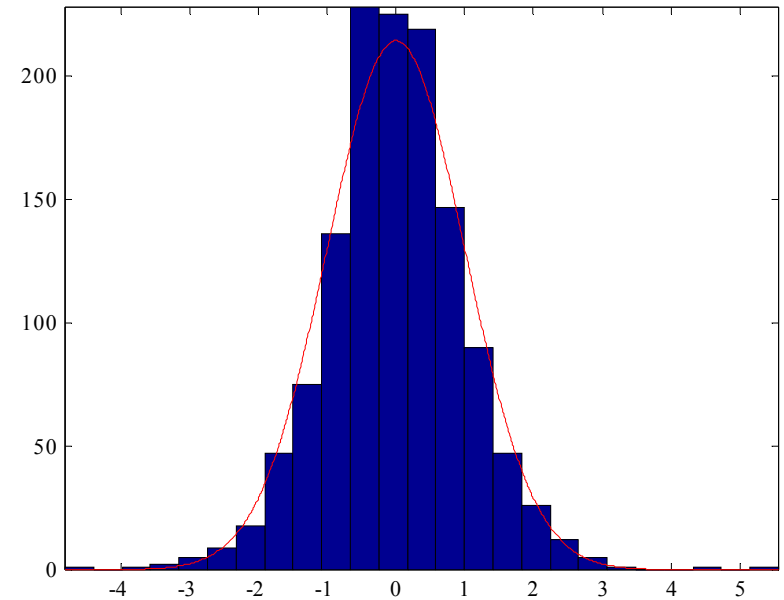
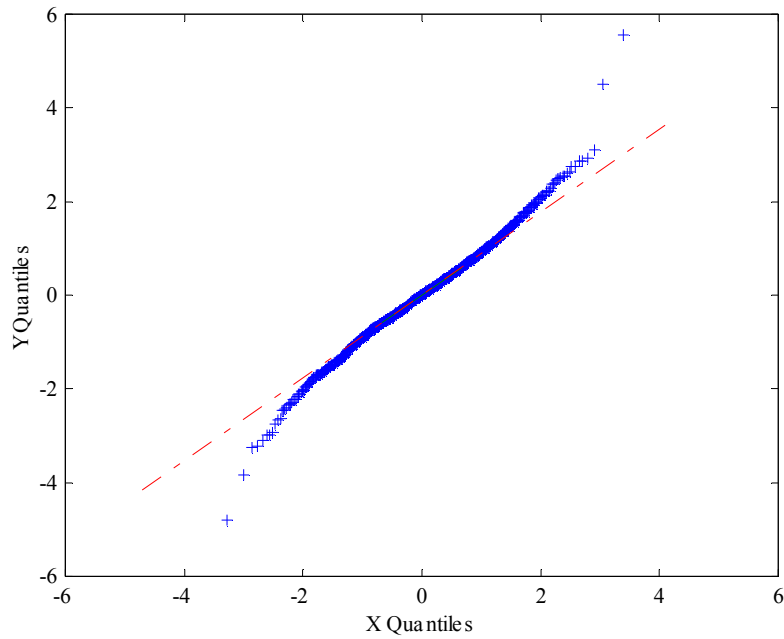
Autocorrelations of residuals



Autocorrelations of abs(residuals)



Normality hypothesis: ARCH(1)



KS Test = 0.12

Figure No. 5

File Edit Tools Window Help

Data file MAX_LENGTH 1296
Tmin Tmax LENGTH = 1296

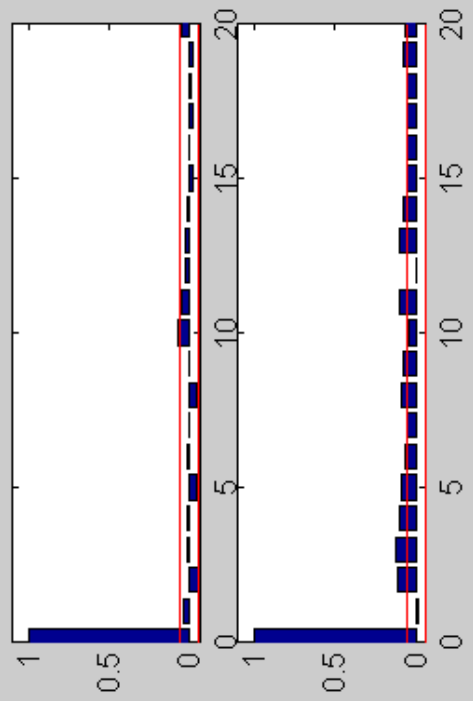
ARCH MODEL

AR (m) FIT

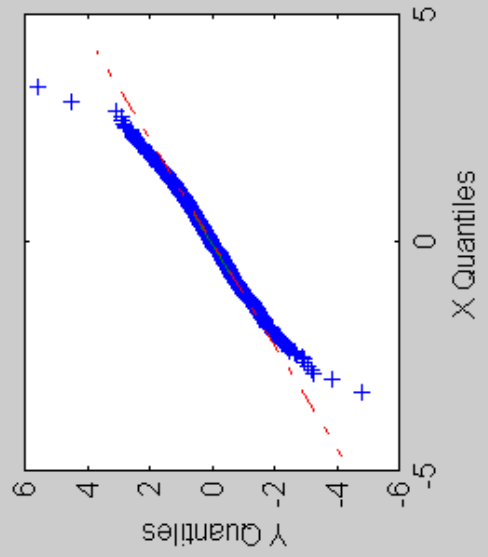
ARCH (q)

phi = 0.1112
kappa = 0.9094
alpha = 0.1117

AUTOCORRELATIONS 20 DAYS



QQPLOT RESIDUALS



AR(1) / GARCH(1,1) for IBEX35

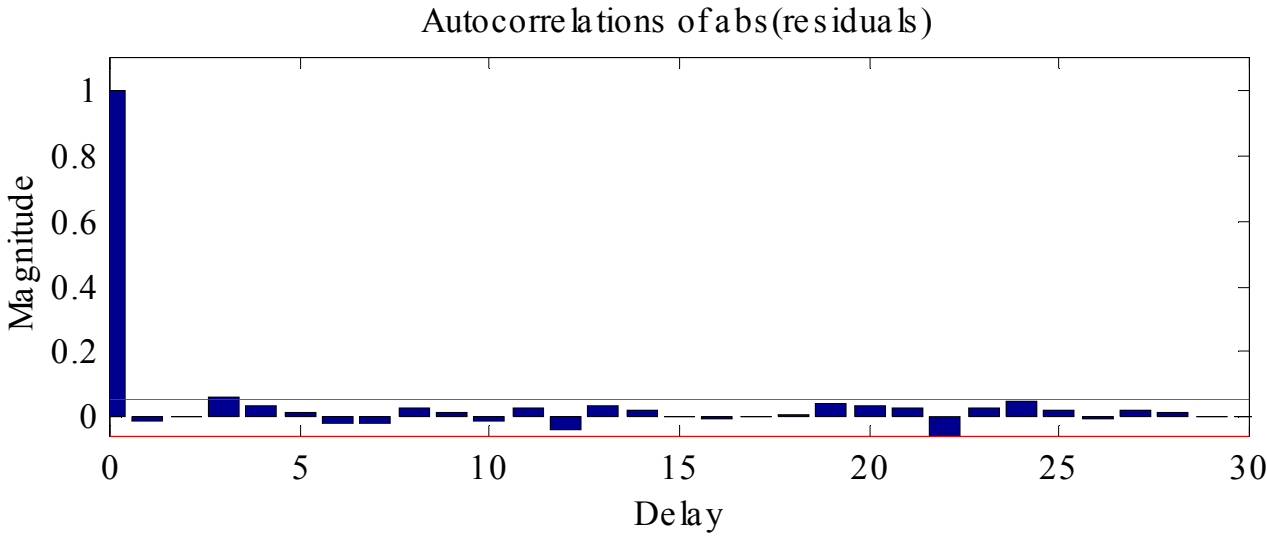
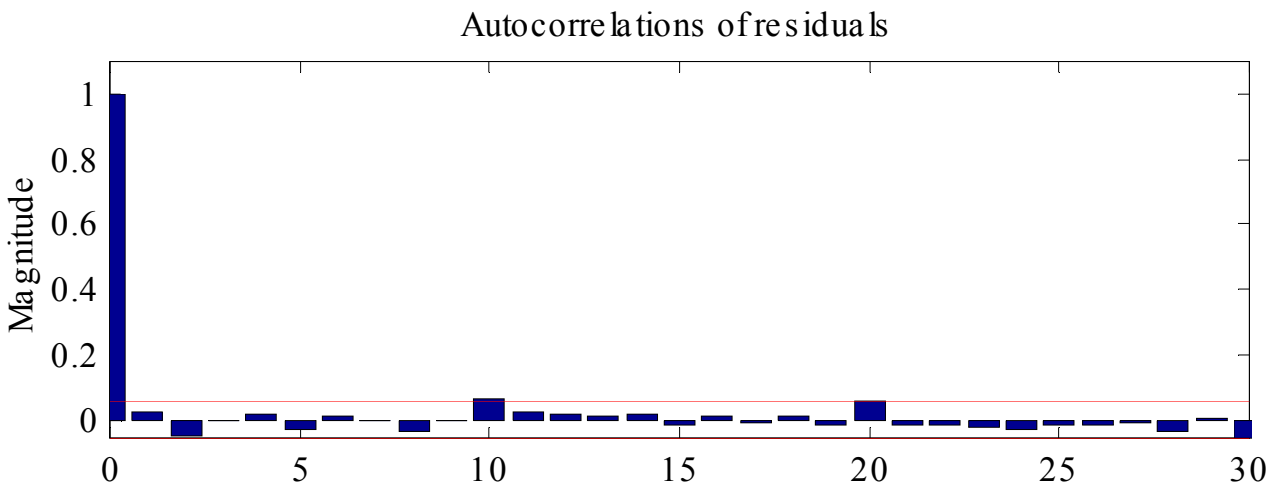
- The maximum-likelihood fit of the time-series IBEX35 yields the model

$$\hat{X}_t = 0.1358\hat{X}_{t-1} + \sigma_t Z_t$$

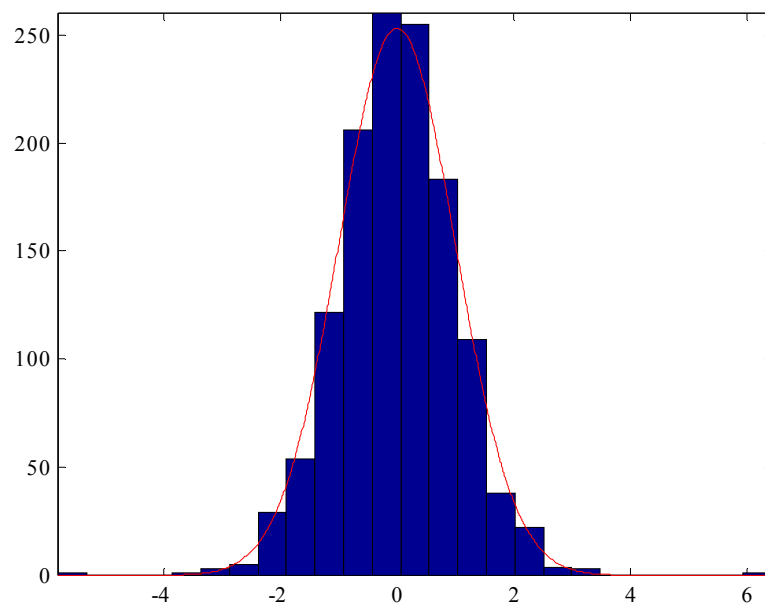
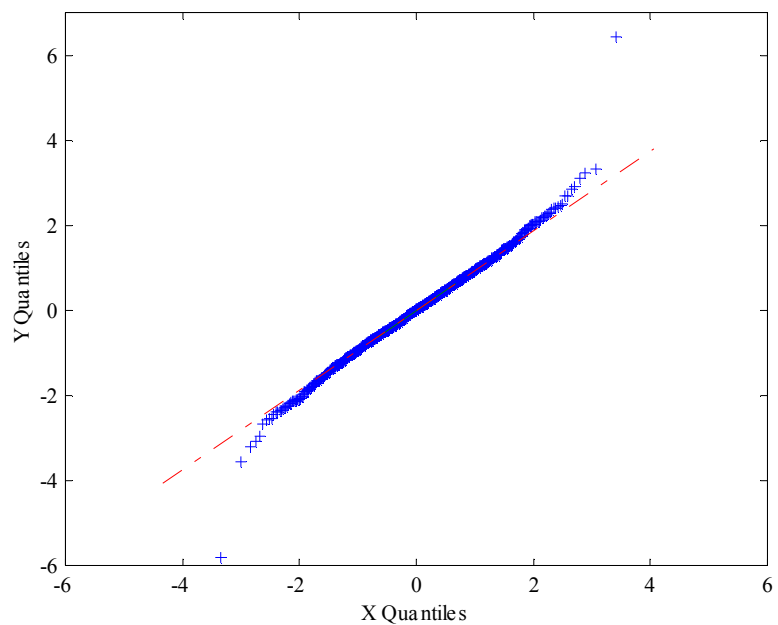
$$\sigma_t^2 = 0.0527 + 0.0755(\hat{X}_{t-1} - 0.1358\hat{X}_{t-2})^2 + 0.8733\sigma_{t-1}^2$$

- The quantities Z_t are assumed to follow a $N(0,1)$ distribution.

Residual correlations: GARCH

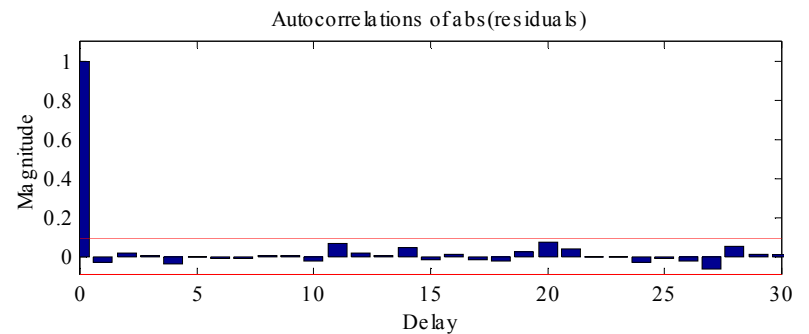
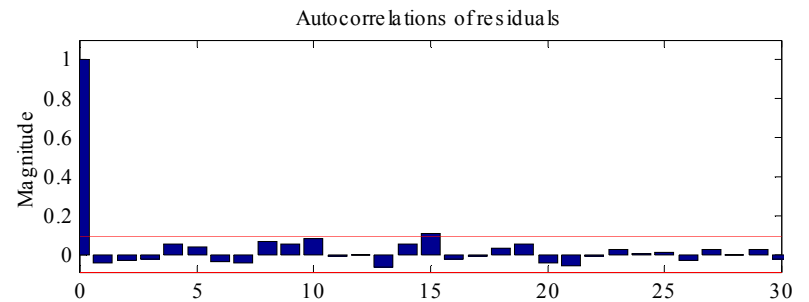
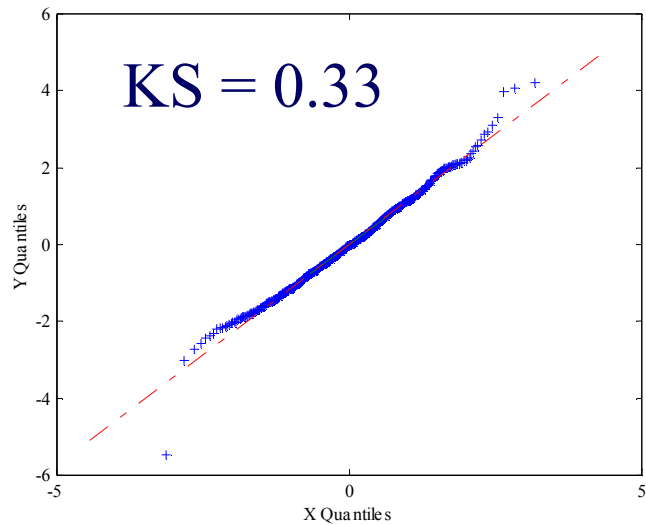
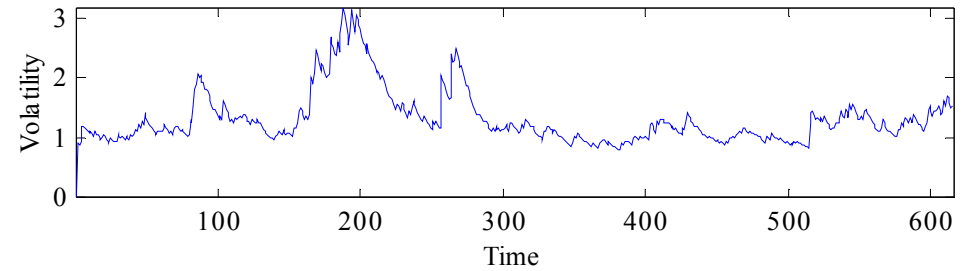
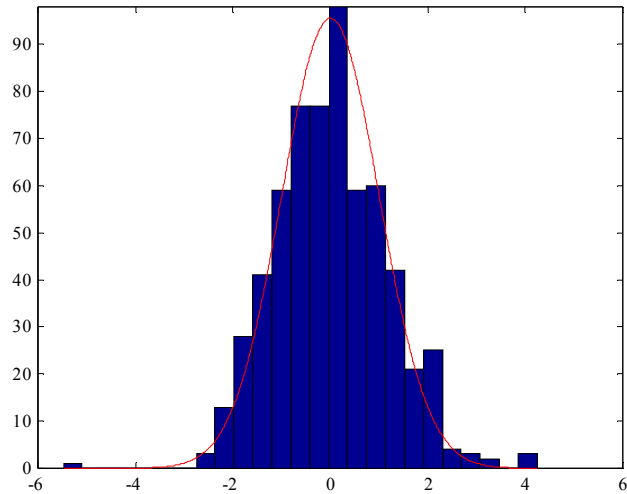


Normality hypothesis: GARCH(1,1)



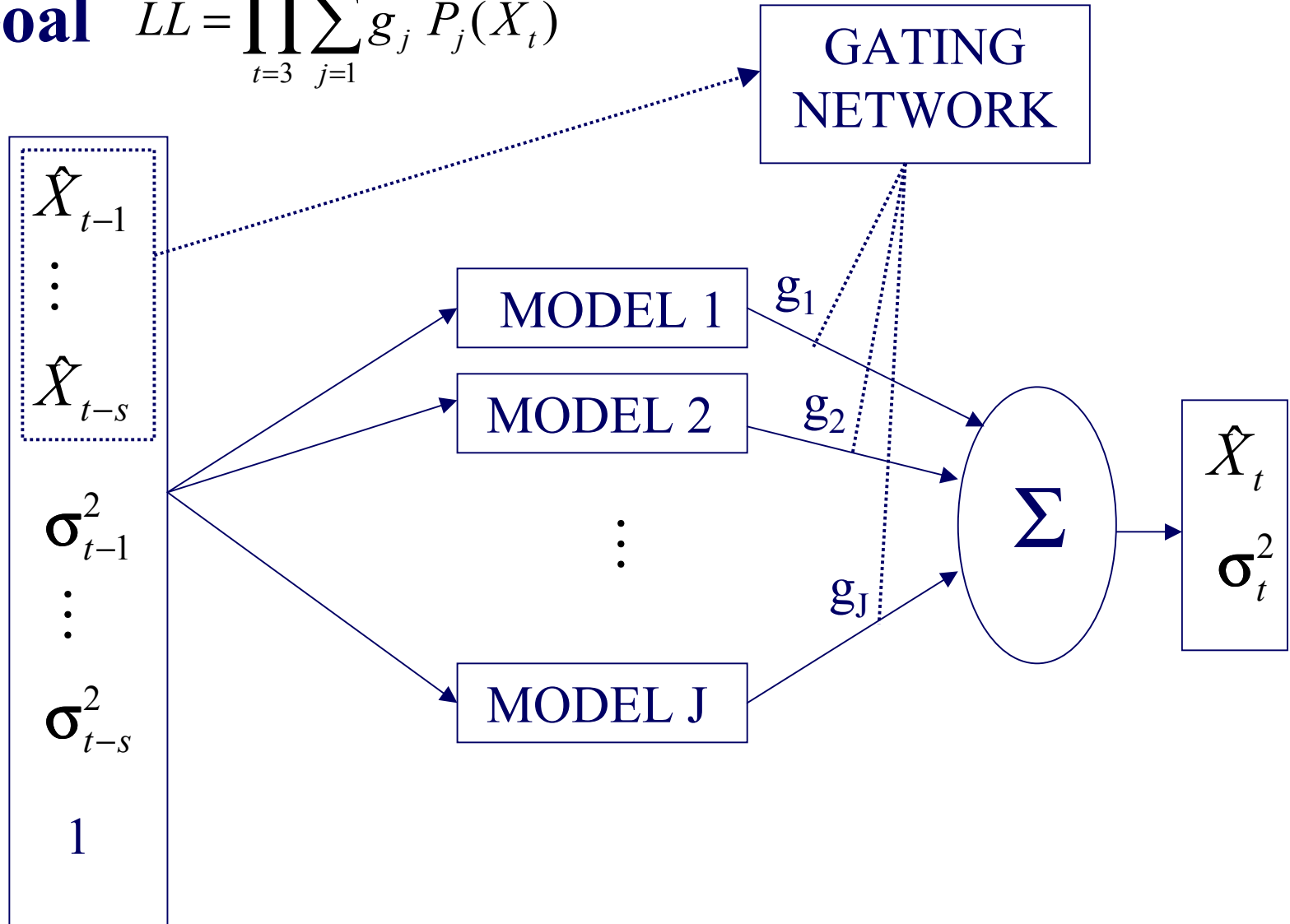
KS Test = 0.56

Test Data



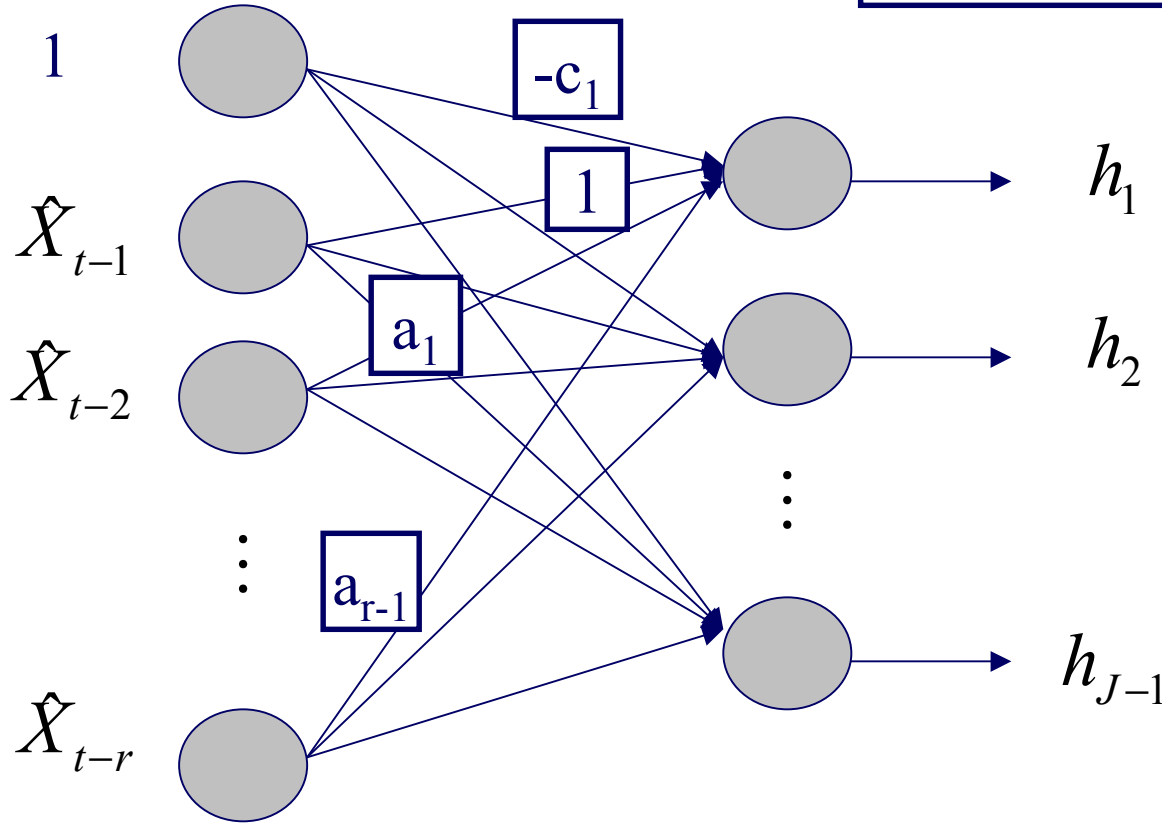
Mixture model

Goal $LL = \prod_{t=3}^T \sum_{j=1}^J g_j P_j(X_t)$



Gating Network

$$h_i = \exp \left[b_i \left(\hat{X}_{t-1} + \sum_{k=1}^{r-1} a_k \hat{X}_{t-k-1} - c_i \right) \right]$$



Probabilities

$$g_i = \frac{h_i}{1 + \sum_{j=1}^{J-1} h_j}; \quad i = 1, 2, \dots, (J-1) \quad g_J = 1 - \sum_{j=1}^{J-1} g_j$$

Mixture of ARCH processes

■ MixARCH $X_t = \phi_{[i]}^+ \cdot \mathbf{X}_t^{[m]} + u_{[i]}(t),$
with probability $g_{[i]}(\mathbf{X}_t^{[r]}, \boldsymbol{\theta}_{[i]})$

➤ The model for the residuals is

$$u_{[i]}(t) = \sigma_{[i]}(t) Z_t$$

$$\sigma_{[i]}^2(t) = \kappa_i + \alpha_i^+ \cdot [u^2]_{[i]}^{[q]}(t)$$

➤ The quantities Z_t are assumed to be $N(0,1)$

MIXARCH for IBEX35

■ The mixture model is

$$\text{Model 1 } \hat{X}_t = 0.0559\hat{X}_{t-1} + \sigma_t Z_t$$

$$\sigma_t^2 = 2.2194 + 0.1976(\hat{X}_{t-1} - 0.0559\hat{X}_{t-2})^2$$

$$\text{Model 2 } \hat{X}_t = 0.1380\hat{X}_{t-1} + \sigma_t Z_t$$

$$\sigma_t^2 = 0.6820 + 0.03821(\hat{X}_{t-1} - 0.1380\hat{X}_{t-2})^2$$

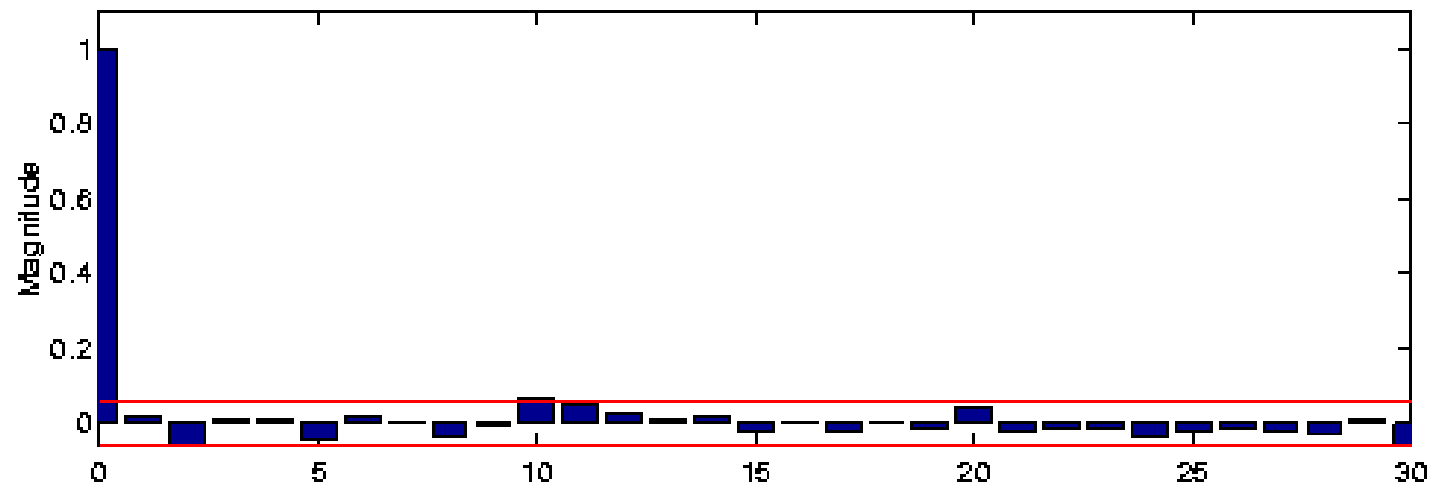
► The probabilities for the mixture are

$$g_{[1]}(X_{t-1}) = \frac{1}{1 + \exp\{-0.6839(X_{t-1} - 2.5155)\}};$$

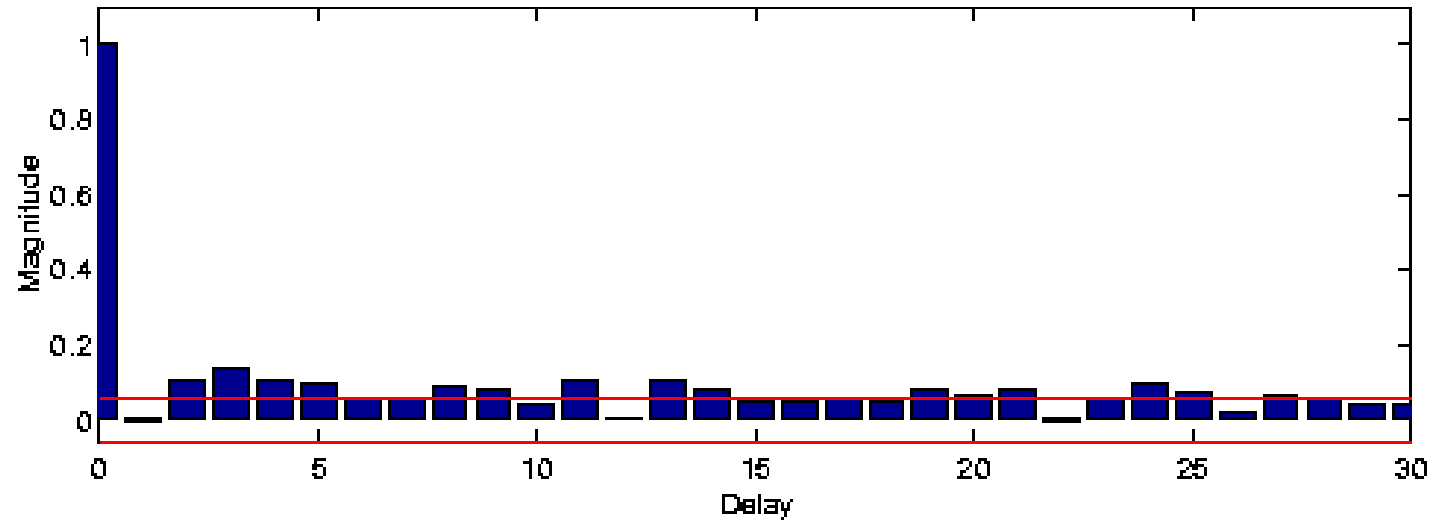
$$g_{[2]}(X_{t-1}) = 1 - g_{[1]}(X_{t-1})$$

Residual correlations: MIXARCH

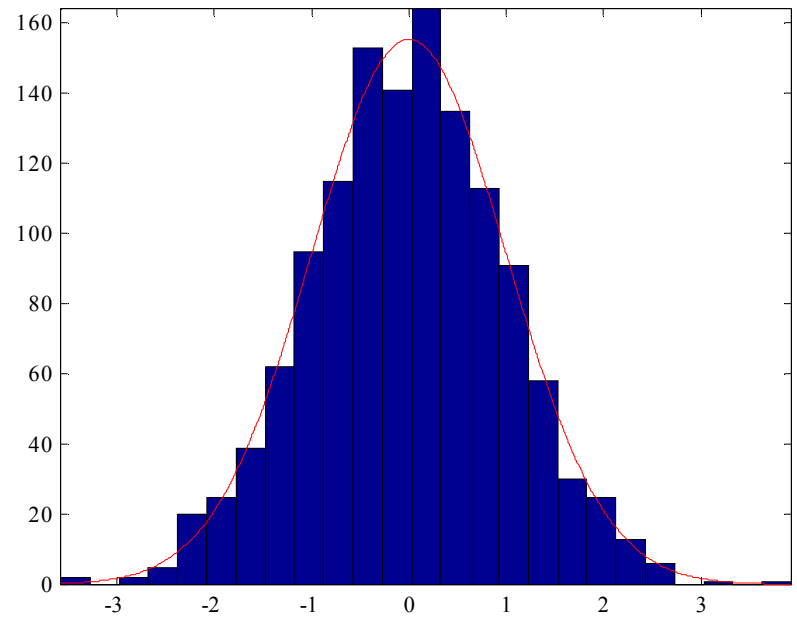
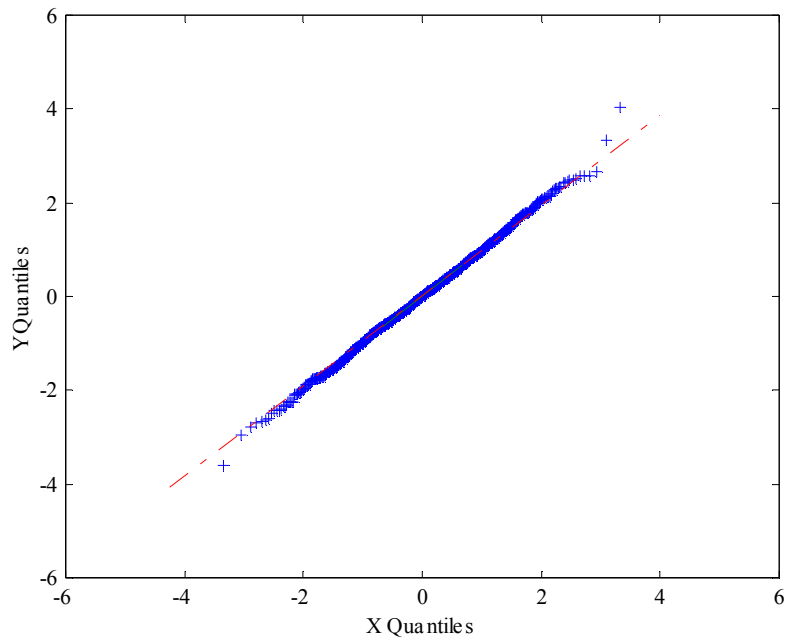
Autocorrelations of residuals



Autocorrelations of abs(residuals)

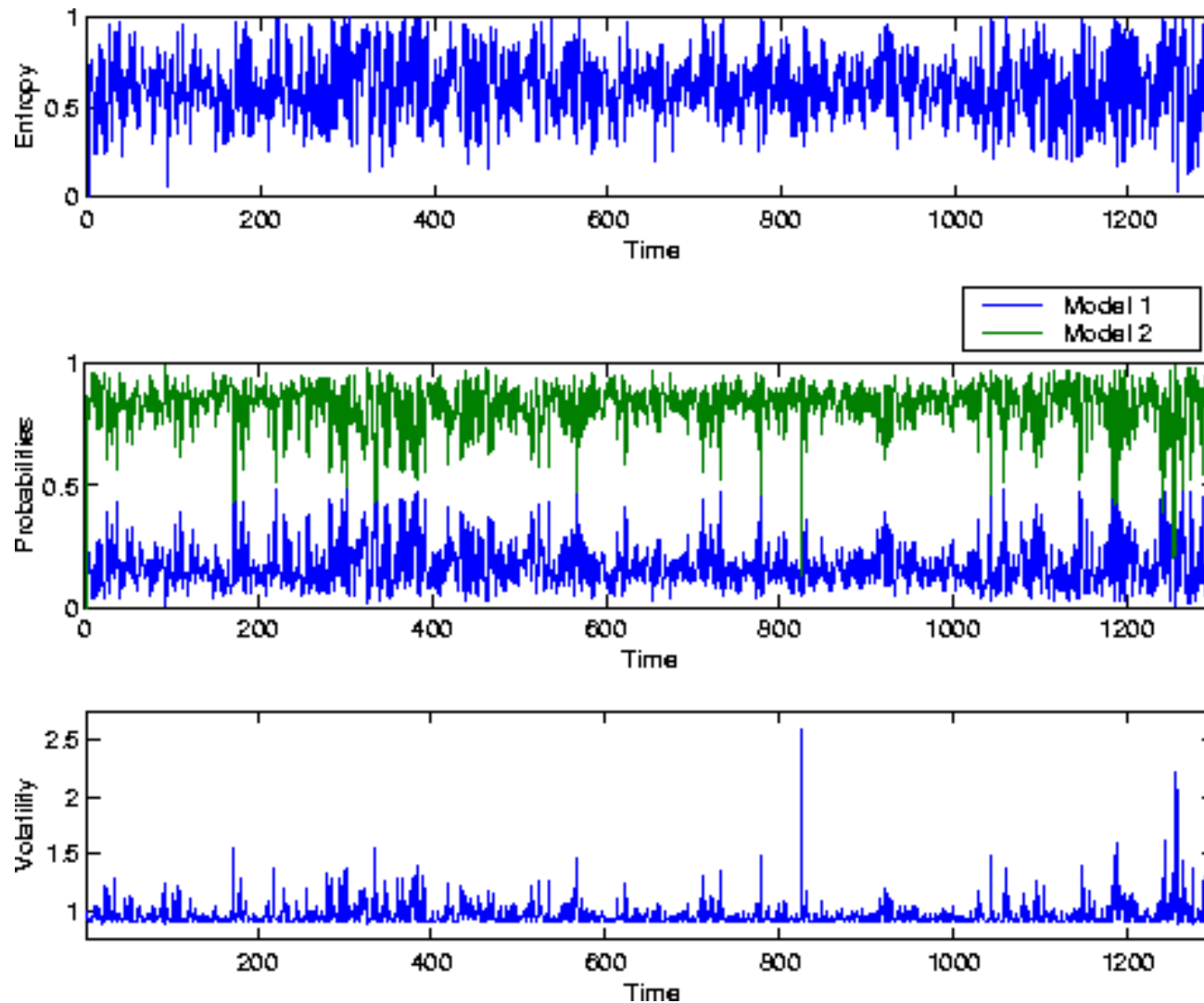


Normality hypothesis: MixARCH(1)



KS Test = 0.83

MIXARCH Model fit



Mixture of GARCH processes

■ MixGARCH $\hat{X}_t = \phi_{[i]}^+ \cdot \hat{X}_t^{[m]} + u_{[i]}(t),$

with probability $g_{[i]}(\mathbf{X}_t^{[r]}, \theta_{[i]})$

➤ The model for the residuals is

$$u_{[i]}(t) = \sigma_{[i]}(t) Z_t$$

$$\sigma_{[i]}^2(t) = \kappa_i + \alpha_i^+ \cdot [u^2]_{[i]}^{[q]}(t) + \beta_i^+ \cdot [\sigma^2]_{[i]}^{[p]}(t)$$

➤ The quantities Z_t are assumed to be $N(0,1)$

MIXGARCH for IBEX35

■ The mixture model is

$$\text{Model 1 } \hat{X}_t = 0.1255\hat{X}_{t-1} + \sigma_t Z_t$$

$$\sigma_t^2 = 0.0156 + 0.0778(\hat{X}_{t-1} - 0.1255\hat{X}_{t-2})^2 + 0.8937\sigma_{t-1}^2$$

$$\text{Model 2 } \hat{X}_t = 0.3314\hat{X}_{t-1} + \sigma_t Z_t$$

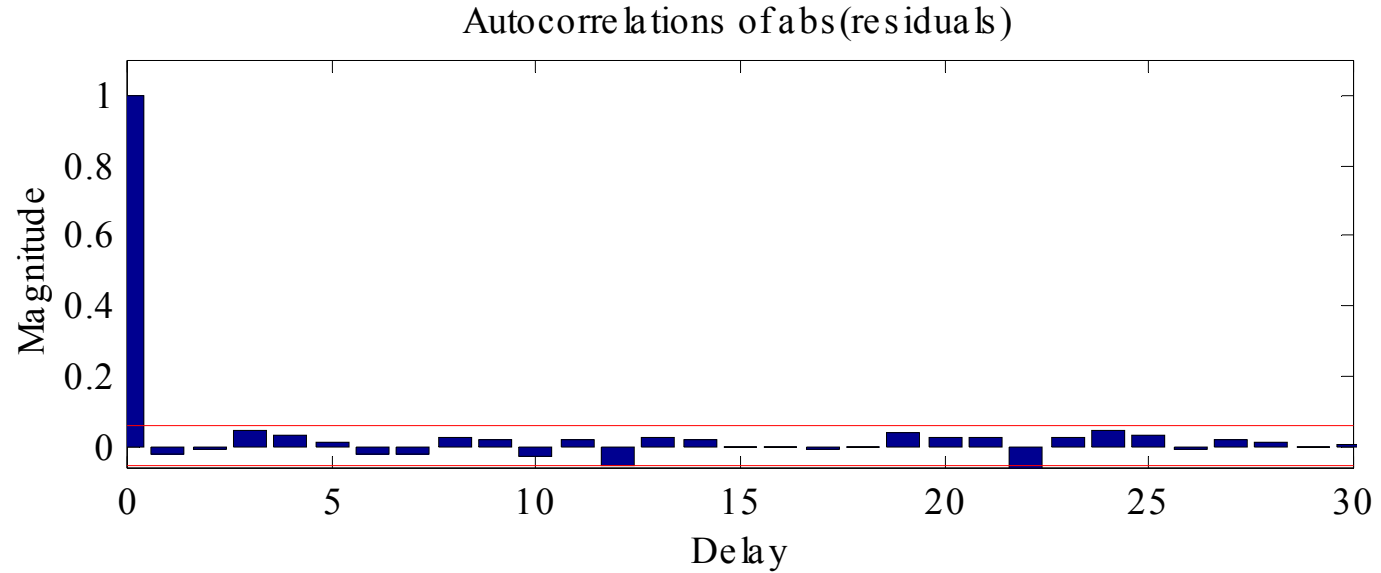
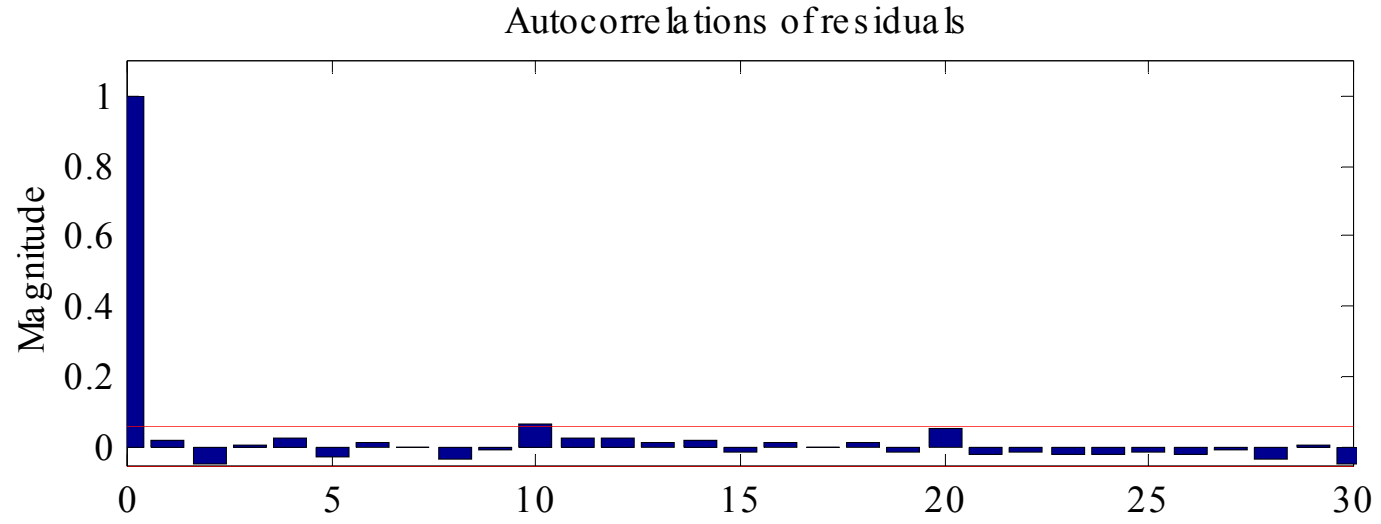
$$\sigma_t^2 = 2.6230 + 0.0000(\hat{X}_{t-1} - 0.3314\hat{X}_{t-2})^2 + 0.0285\sigma_{t-1}^2$$

► The probabilities for the mixture are

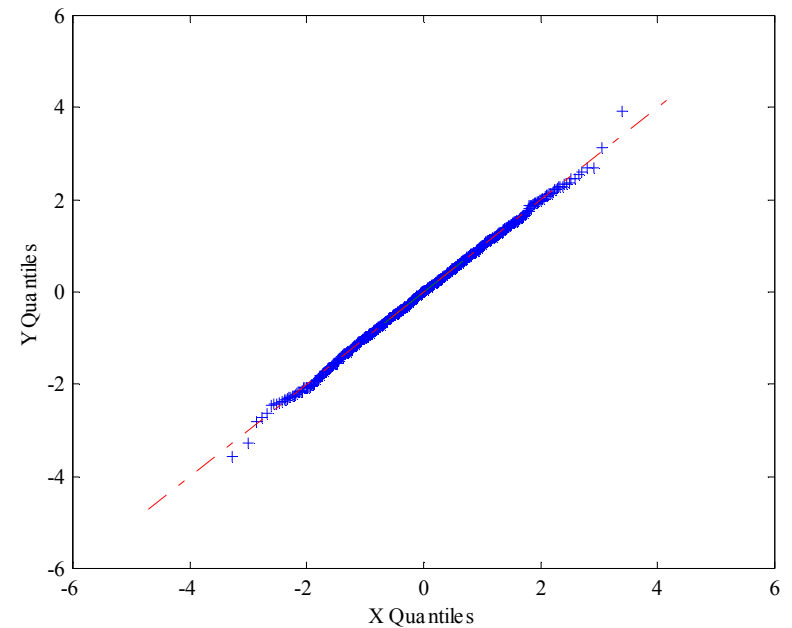
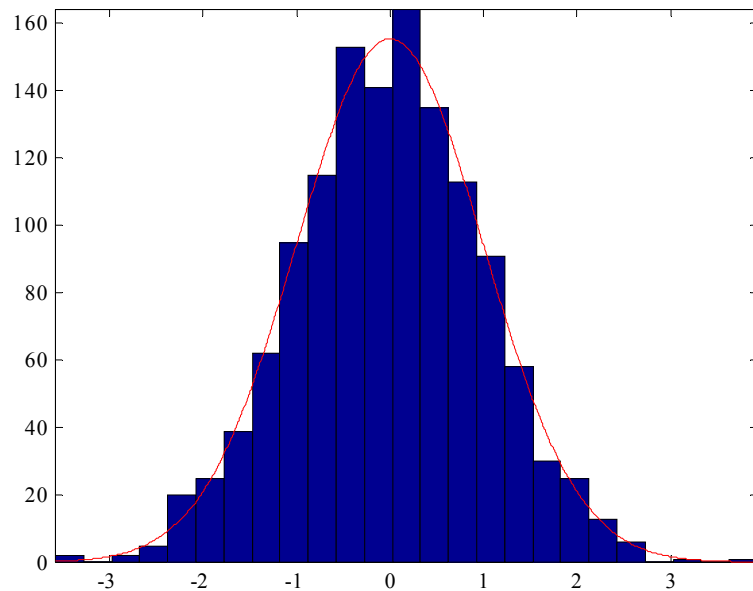
$$g_{[1]}(X_{t-1}) = \frac{1}{1 + \exp\{0.5418(\hat{X}_{t-1} - 4.8710)\}};$$

$$g_{[2]}(X_{t-1}) = 1 - g_{[1]}(X_{t-1})$$

Residual correlations: MIXGARCH

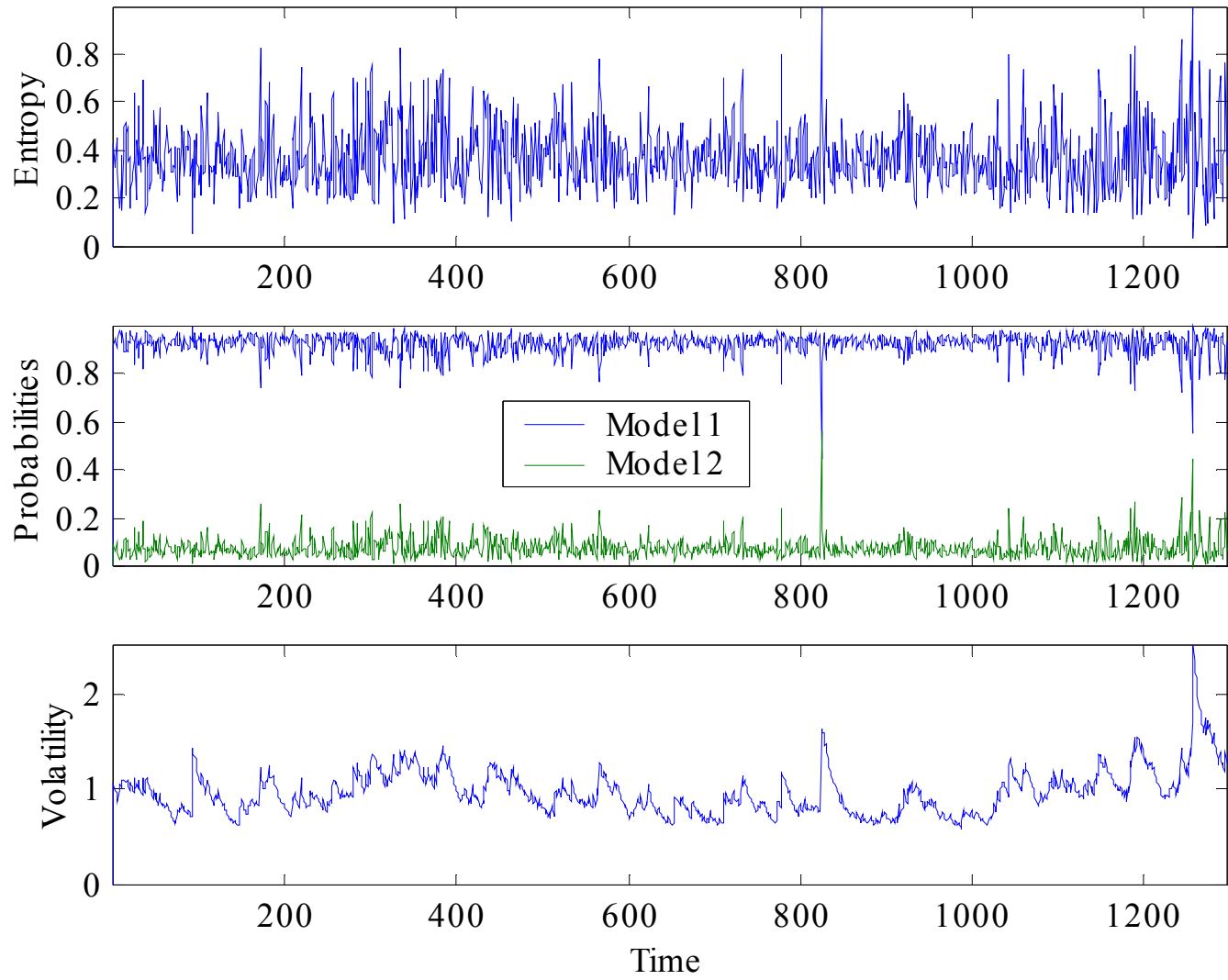


Normality hypothesis: MIXGARCH

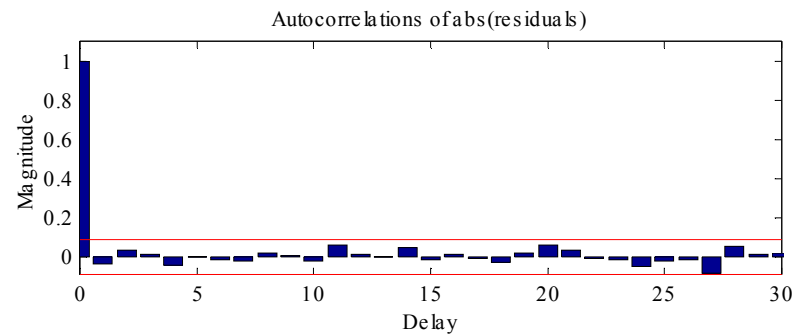
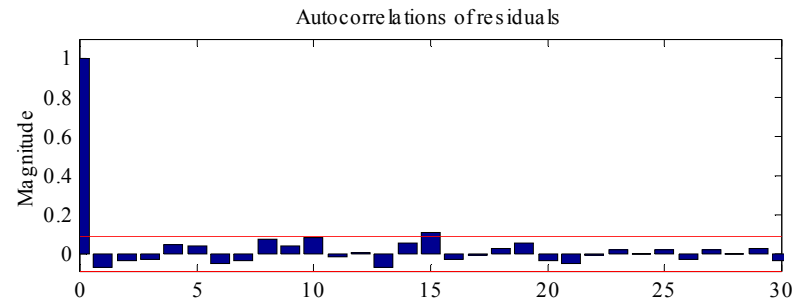
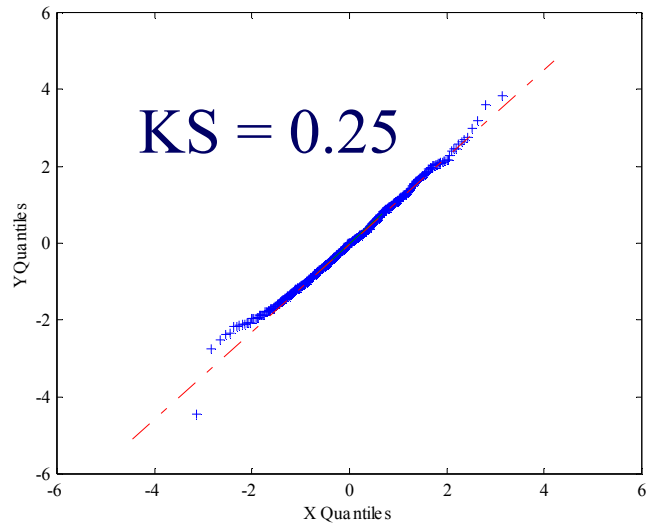
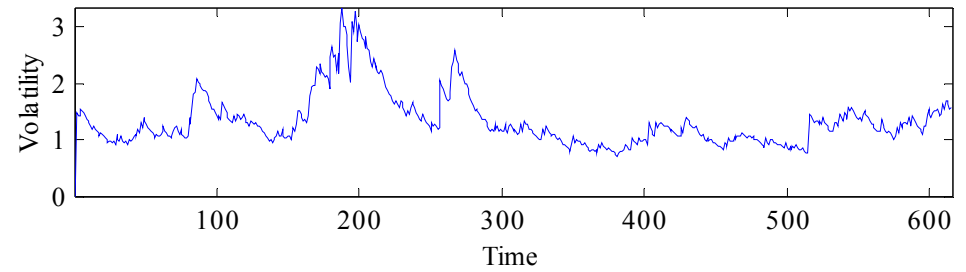
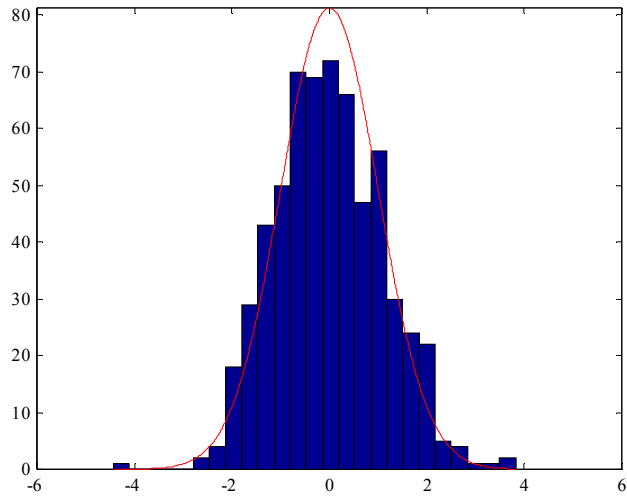


KS test = 0.95

MIXGARCH Model fit



Test Data



Conclusions

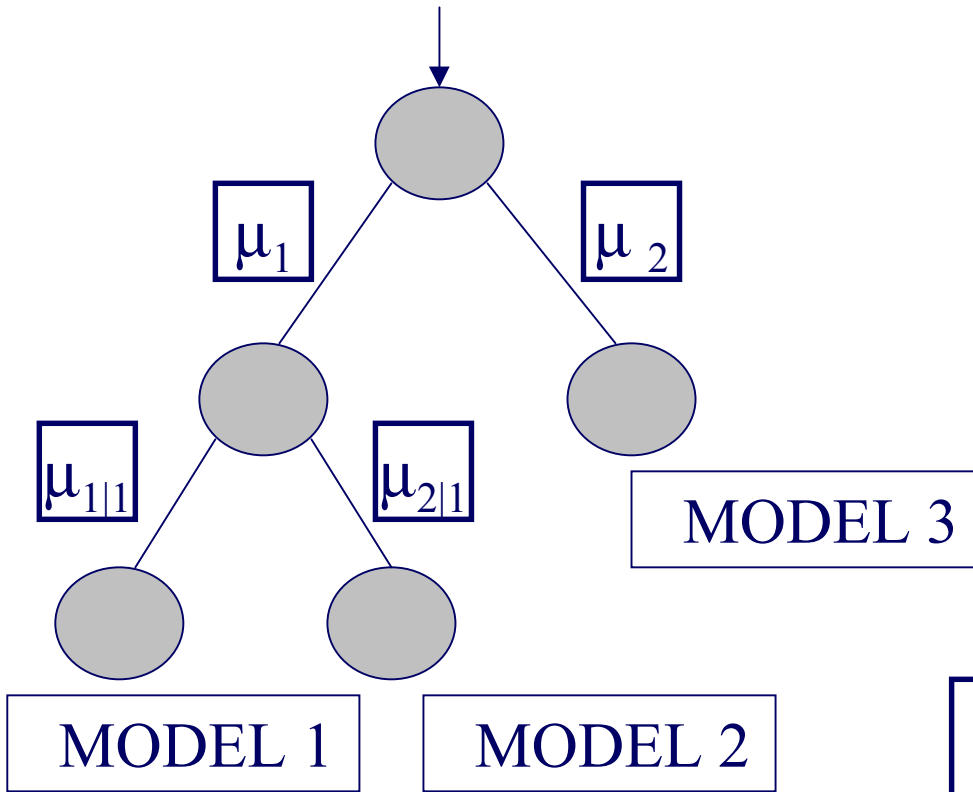


- **Autoregressive mixture models** can be used to improve the modeling of financial time series
 - Model is piecewise-linear.
 - Memory effects.
 - Extreme events.
- **Difficulties:**
 - Parsimony? (Convex linear combination of simple base models)
 - Constrained optimization of LL function.
 - EM algorithm is not practical.

Perspectives:

Decision trees for TS analysis

Input = Vector of Delayed values



$$\mu_1 = \frac{\exp \left[b_1 \left(\hat{X}_{t-1} + \sum_{k=1}^{r-1} a_{1k} \hat{X}_{t-k-1} - c_1 \right) \right]}{1 + \exp \left[b_1 \left(\hat{X}_{t-1} + \sum_{k=1}^{r-1} a_{1k} \hat{X}_{t-k-1} - c_1 \right) \right]}$$

$$\mu_2 = 1 - \mu_1$$

$$\mu_{1|1} = \frac{\exp \left[b_2 \left(\hat{X}_{t-1} + \sum_{k=1}^{r-1} a_{2k} \hat{X}_{t-k-1} - c_2 \right) \right]}{1 + \exp \left[b_2 \left(\hat{X}_{t-1} + \sum_{k=1}^{r-1} a_{2k} \hat{X}_{t-k-1} - c_2 \right) \right]}$$

$$\mu_{2|1} = 1 - \mu_{1|1}$$

Model	1	$\mu_{11} = \mu_{1 1} \mu_1;$
Model	2	$\mu_{12} = \mu_{2 1} \mu_1;$
Model	3	μ_2

Acknowledgements



- **RiskLab Madrid.**
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